國立臺灣海洋大學河海工程學系 2026 工程數學(二)2AB 班課後練習 09 參考解答

系級：______________ 學號：____________ 姓名：___________

1. 給一週期函數 $f(x)=x^2$，$-1<x<1$ 且 $f(x)=f(x+2)$

   (1) 試求其傅立葉級數展開。

   (2) $\sum_{n=1}^{\infty}\frac{1}{n^2}=?$

   (3) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=?$

   (4) $\sum_{n=1}^{\infty}\frac{1}{n^4}=?$

   (5) $\sum_{n=1}^{\infty}\frac{1}{n^6}=?$

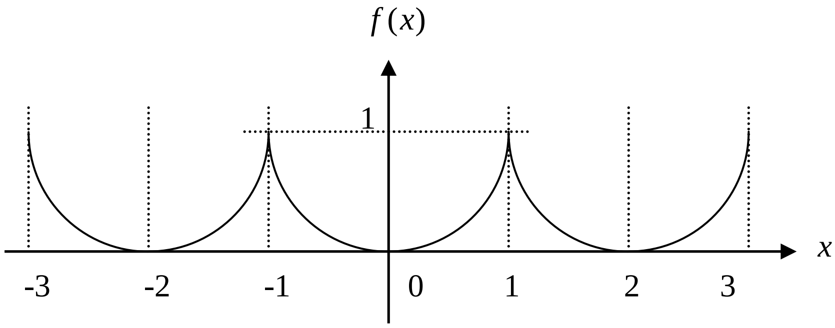
   (6) $\sum_{n=1}^{\infty}\frac{1}{n^8}=?$

2. 試求 $f(x)=\begin{cases}1, 0\le x<1\\ 2, 1<x\le 2\end{cases}$ 之傅立葉全幅展開、半幅正弦展開與半幅餘弦展開並畫出相對應之圖形。

3. 試求 $f(x)=x^2+x$ 就其在區間 $(0, 2)$ 上之全幅展開、半幅正弦展開與半幅餘弦展開並畫出相對應之圖形。

參考解答:

1.



(1) $f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos\frac{2n\pi x}{T}+\sum_{n=1}^{\infty}b_n\sin\frac{2n\pi x}{T}$ （已知 $T=2$，$b_n=0$）

$\Rightarrow f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos n\pi x$

$$a_0=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}f(x)\,dx=\frac{1}{2}\int_{-1}^{1}x^2\,dx=\frac{1}{3}$$

$$a_n=\frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}f(x)\cos n\pi x\,dx=\int_{-1}^{1}x^2\cos n\pi x\,dx=\frac{4(-1)^n}{n^2\pi^2}$$

$$\therefore\quad f(x)=\frac{1}{3}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x$$

(2) $f(1)=1^2=\frac{1}{3}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi \quad\Rightarrow\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$

(3) $f(0)=0^2=\frac{1}{3}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\quad\Rightarrow\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{12}$

(4) 應用 Parseval 定理: $\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}f^2(x)\,dx=a_0^2+\frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)$

可得 $\frac{1}{2}\int_{-1}^{1}x^4\,dx=\frac{1}{9}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{16}{n^4\pi^4}\quad\Rightarrow\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}$

(5) 由傅立葉級數展開可知 $f(x)=x^2=\frac{1}{3}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x$

積分後可得

$\frac{1}{3}x^3=\frac{1}{3}x+\frac{4}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin n\pi x+c\quad\Rightarrow\frac{1}{3}x^3-\frac{1}{3}x=\frac{4}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin n\pi x+c$

當 $x=1$ 代入，可得 $c=0$

應用 Parseval 定理可得 $\sum_{n=1}^{\infty}\frac{16}{\pi^6n^6}=\int_{-1}^{1}(\frac{1}{3}x^3-\frac{1}{3}x)^2\,dx=\frac{16}{945}$

$$\Rightarrow\sum_{n=1}^{\infty}\frac{1}{n^6}=\frac{\pi^6}{945}$$

(6) 再積分一次後可得

$$\frac{1}{12}x^4-\frac{1}{6}x^2=-\frac{4}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\cos n\pi x+c$$

當 $x=1$ 代入，可得 $\frac{1}{12}-\frac{1}{6}=-\frac{4}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}+c \quad \Rightarrow c=-\frac{7}{180}$

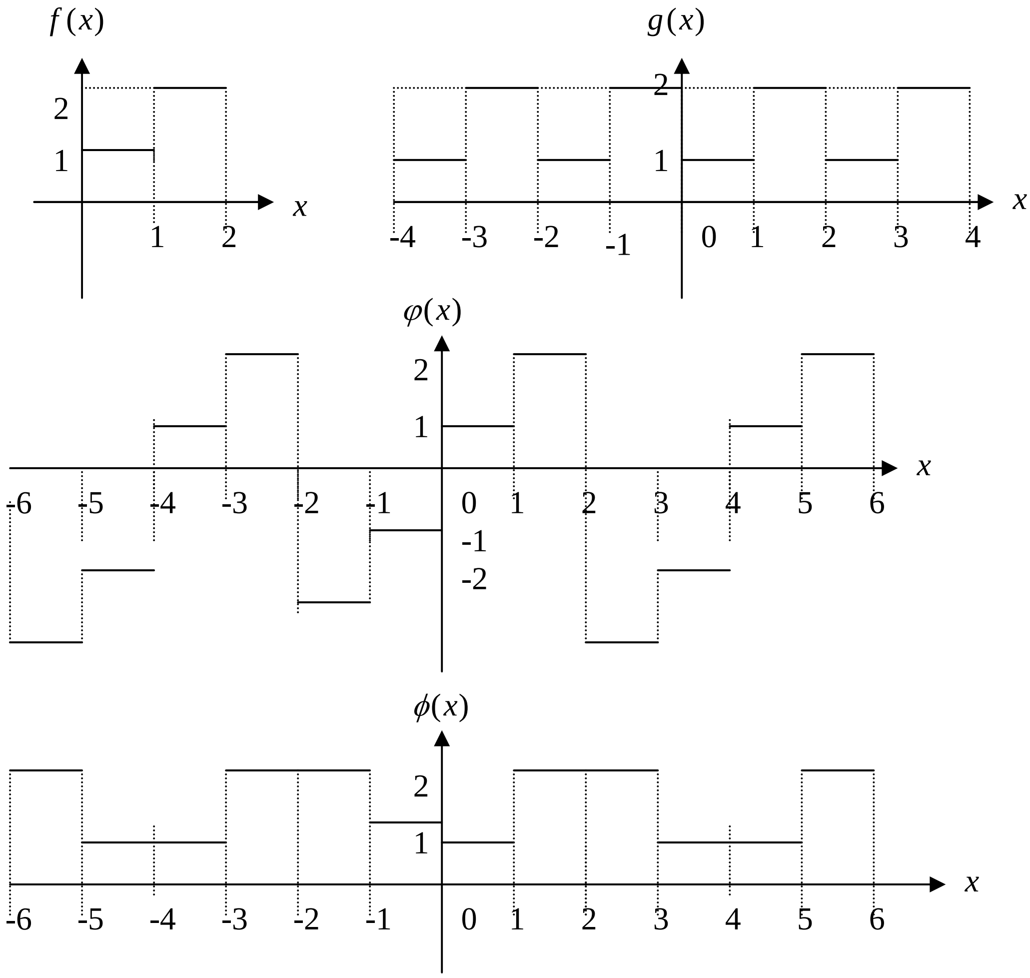
$$\therefore\ \frac{1}{12}x^4-\frac{1}{6}x^2=-\frac{7}{180}+\frac{4}{\pi^4}\sum_{n=1}^{\infty}-\frac{(-1)^n}{n^4}\cos n\pi x$$

應用 Parseval 定理可得 $\frac{1}{2}\int_{-1}^{1}(\frac{1}{12}x^4-\frac{1}{6}x^2)^2\,dx=(-\frac{7}{180})^2+\frac{1}{2}\sum_{n=1}^{\infty}\frac{16}{\pi^8 n^8}$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{16}{\pi^8 n^8}=\frac{8}{4725}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^8}=\frac{\pi^8}{9450}$$

2.



全幅展開

[0, 2] 為完整週期 $\Rightarrow T=2$

$$g(x)=a_0+\sum_{n=1}^{\infty}a_n\cos\frac{2n\pi x}{T}+b_n\sin\frac{2n\pi x}{T}=a_0+\sum_{n=1}^{\infty}a_n\cos(n\pi x)+b_n\sin(n\pi x)$$

$$a_0 = \frac{1}{T}\int_0^T f(x)\,dx = \frac{1}{2}\int_0^2 f(x)\,dx = \frac{1}{2}(\int_0^1 1\,dx + \int_1^2 2\,dx) = \frac{3}{2}$$

$$a_n = \frac{2}{T}\int_0^T f(x)\cdot\cos\frac{2n\pi x}{T}\,dx = \int_0^2 f(x)\cdot\cos(n\pi x)\,dx$$

$$= \int_0^1 \cos(n\pi x)\,dx + \int_1^2 2\cdot\cos(n\pi x)\,dx = 0$$

$$b_n = \frac{2}{T}\int_0^T f(x)\cdot\sin\frac{2n\pi x}{T}\,dx = \int_0^2 f(x)\cdot\sin(n\pi x)\,dx$$

$$= \int_0^1 \sin(n\pi x)\,dx + \int_1^2 2\sin(n\pi x)\,dx$$

$$= \frac{1}{n\pi}[(-1)^n - 1]$$

$$g(x) = \frac{3}{2} + \sum_{n=1}^{\infty}\frac{[(-1)^n - 1]}{n\pi}\sin n\pi x$$

半幅正弦展開

$[0, 2]$ 為一半週期 $\Rightarrow T = 4$

$$\varphi(x) = \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi x}{T}$$

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cdot\sin\frac{2n\pi x}{T}\,dx = \int_0^2 f(x)\cdot\sin\frac{n\pi x}{2}\,dx$$

$$= \int_0^1 \sin\frac{n\pi x}{2}\,dx + \int_1^2 2\sin\frac{n\pi x}{2}\,dx$$

$$= \frac{2}{n\pi}[1 - 2(-1)^n + \cos(\frac{n\pi}{2})]$$

$$\varphi(x) = \sum_{n=1}^{\infty}\frac{2}{n\pi}[1 - 2(-1)^n + \cos(\frac{n\pi}{2})]\sin\frac{n\pi x}{2}$$

半幅餘弦展開

$[0, 2]$ 為一半週期 $\Rightarrow T = 4$

$$\phi(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{T}$$

$$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\,dx = \frac{1}{2}\int_0^2 f(x)\,dx = \frac{1}{2}(\int_0^1 1\,dx + \int_1^2 2\,dx) = \frac{3}{2}$$

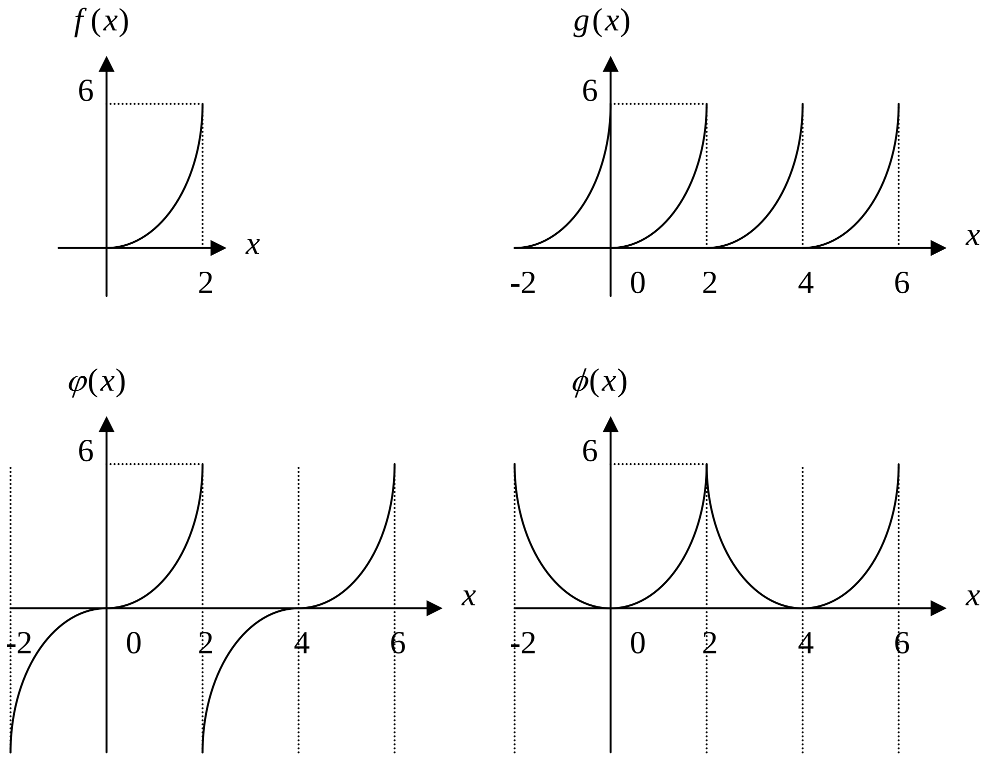
$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cdot\cos\frac{2n\pi x}{T}\,dx = \int_0^2 f(x)\cdot\cos\frac{n\pi x}{2}\,dx$$

$$= \int_0^1 \cos\frac{n\pi x}{2}\,dx + \int_1^2 2\cos\frac{n\pi x}{2}\,dx$$

$$= -\frac{2}{n\pi}\sin\frac{n\pi}{2}$$

$$\phi(x) = \frac{3}{2} - \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin\frac{n\pi}{2}\cos\frac{n\pi x}{2}$$

3.



全幅展開

(0, 2) 為完整週期 $\Rightarrow T = 2$

$$g(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{2n\pi x}{T} + b_n\sin\frac{2n\pi x}{T} = a_0 + \sum_{n=1}^{\infty} a_n\cos n\pi x + b_n\sin n\pi x$$

$$a_0 = \frac{1}{T}\int_0^T f(x)\,dx = \frac{1}{2}\int_0^2 (x + x^2)\,dx = \frac{7}{3}$$

$$a_n = \frac{2}{T}\int_0^T f(x)\cdot\cos\frac{2n\pi x}{T}\,dx = \int_0^2 (x + x^2)\cdot\cos n\pi x\,dx = \frac{4}{n^2\pi^2}$$

$$b_n = \frac{2}{T}\int_0^T f(x)\cdot\sin\frac{2n\pi x}{T}\,dx = \int_0^2 (x + x^2)\cdot\sin n\pi x\,dx = -\frac{6}{n\pi}$$

$$g(x) = \frac{7}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\cos n\pi x - \frac{6}{n\pi}\sin n\pi x$$

半幅正弦展開

(0, 2) 為一半週期 $\Rightarrow T=4$

$$\varphi(x)=\sum_{n=1}^{\infty} b_n \sin\frac{2n\pi x}{T}$$

$$b_n=\frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cdot\sin\frac{2n\pi x}{T}dx=\int_0^2 (x+x^2)\cdot\sin n\pi x\,dx$$

$$=\frac{12}{n\pi}(-1)^{n+1}+\frac{16}{n^3\pi^3}[(-1)^n-1]$$

$$\varphi(x)=\sum_{n=1}^{\infty}[\frac{12}{n\pi}(-1)^{n+1}+\frac{16}{n^3\pi^3}((-1)^n-1)]\sin\frac{n\pi x}{2}$$

半幅餘弦展開

(0, 2) 為一半週期 $\Rightarrow T=4$

$$\phi(x)=a_0+\sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{T}$$

$$a_0=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\,dx=\frac{1}{2}\int_0^2 (x+x^2)\,dx=\frac{7}{3}$$

$$a_n=\frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cdot\cos\frac{2n\pi x}{T}dx=\int_0^2 (x+x^2)\cdot\cos\frac{n\pi x}{2}dx=\frac{20\cdot(-1)^n-4}{n^2\pi^2}$$

$$\phi(x)=\frac{7}{3}+\sum_{n=1}^{\infty}\frac{20\cdot(-1)^n-4}{n^2\pi^2}\cos\frac{n\pi x}{2}$$