國立臺灣海洋大學河海工程學系 2026 工程數學(二)2AB 班課後練習 05 參考解答

系級：____________ 學號：____________ 姓名：____________

1. 令 $\vec{F}=P\vec{i}+Q\vec{j}+R\vec{k}$, $\vec{G}=U\vec{i}+V\vec{j}+W\vec{k}$ ，試證明:

   (1) $\nabla\cdot(\vec{F}\times\vec{G})=\vec{G}\cdot(\nabla\times\vec{F})-\vec{F}\cdot(\nabla\times\vec{G})$

   (2) $\nabla(\vec{F}\cdot\vec{G})=(\vec{F}\cdot\nabla)\vec{G}+(\vec{G}\cdot\nabla)\vec{F}+\vec{F}\times(\nabla\times\vec{G})+\vec{G}\times(\nabla\times\vec{F})$

   (3) $\nabla\times(\nabla\times\vec{F})=\nabla(\nabla\cdot\vec{F})-\nabla^2\vec{F}$

2. 給一向量函數 $\vec{F}(x,y,z)=F_1\vec{i}+F_2\vec{j}+F_3\vec{k}$ 又

   $$curl\vec{F}=\nabla\times\vec{F}=(-4y^3z^6-4x^5y^2)\,\vec{i}-4z^3\;\vec{j}+(20x^4y^2z-3x^2y^2)\,\vec{k}$$

   $$div\vec{F}=\nabla\cdot\vec{F}=2xy^3+8x^5yz-6y^4z^5$$

   試找出可能之 F_1,F_2 與 F_3。(Hint: 此為非唯一解，試由觀察來得可能之解)

3. 試求 $\int_C x^2y\,ds$ 而 C 之路徑為 $\vec{r}(t)=a\cos t\,\vec{i}+a\sin t\,\vec{j}$ 且 $t\in[0,\pi]$

4. 試求 $\int_C (xy+z)\,ds$, 其中 C 為球面 $x^2+y^2+z^2=1$ 與平面 $3z=4y$ 之交線。

5. 試求 $\int_C y^2dx-xy\,dy$ 而 C 為 $y=3x-x^2$ 且 $x\in[0,3]$

6. 試求場 $\vec{F}=xy\vec{i}+(3x-y^2)\,\vec{j}$，試問自 $(5,6)$ 至 $(3,3)$ 之直線線積分值，自 $(5,6)$ 經 $(5,3)$ 至 $(3,3)$ 之折線線積分值，並問 $\vec{F}$ 是否為保守場。

7. 試計算線積分 $\int_C \vec{F}\cdot d\vec{r}$ ，其中 $\vec{F}=6x^2\,\vec{i}-2x\,\vec{j}$ 路徑 C 為由 $(5,4)\to(1,3)\to(0,1)\to(5,1)$ 之三條直線所組成。

參考解答:

1. (1) $\nabla\cdot(\vec{F}\times\vec{G})=\vec{G}\cdot(\nabla\times\vec{F})-\vec{F}\cdot(\nabla\times\vec{G})$

$$\nabla\cdot(\vec{F}\times\vec{G})=(\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k})\cdot[(QW-RV)\vec{i}+(RU-PW)\vec{j}+(PV-QU)\vec{k}]$$
$$=\frac{\partial(QW-RV)}{\partial x}+\frac{\partial(RU-PW)}{\partial y}+\frac{\partial(PV-QU)}{\partial z}$$
$$=\frac{\partial Q}{\partial x}W+Q\frac{\partial W}{\partial x}-\frac{\partial R}{\partial x}V-R\frac{\partial V}{\partial x}+\frac{\partial R}{\partial y}U+R\frac{\partial U}{\partial y}-\frac{\partial P}{\partial y}W-P\frac{\partial W}{\partial y}$$
$$+\frac{\partial P}{\partial z}V+P\frac{\partial V}{\partial z}-\frac{\partial Q}{\partial z}U-Q\frac{\partial U}{\partial z}$$
$$=P(\frac{\partial V}{\partial z}-\frac{\partial W}{\partial y})+Q(\frac{\partial W}{\partial x}-\frac{\partial U}{\partial z})+R(\frac{\partial U}{\partial y}-\frac{\partial V}{\partial x})$$
$$+U(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z})+V(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x})+W(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})$$
$$=(P\vec{i}+Q\vec{j}+R\vec{k})\cdot[(\frac{\partial V}{\partial z}-\frac{\partial W}{\partial y})\vec{i}+(\frac{\partial W}{\partial x}-\frac{\partial U}{\partial z})\vec{j}+(\frac{\partial U}{\partial y}-\frac{\partial V}{\partial x})\vec{k}]$$
$$+(U\vec{i}+V\vec{j}+W\vec{k})\cdot[(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z})\vec{i}+(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x})\vec{j}+(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})\vec{k}]$$
$$=\vec{F}\cdot(-\nabla\times\vec{G})+\vec{G}\cdot(\nabla\times\vec{F})$$
$$=\vec{G}\cdot(\nabla\times\vec{F})-\vec{F}\cdot(\nabla\times\vec{G})$$

(2) $\nabla(\vec{F}\cdot\vec{G})=(\vec{F}\cdot\nabla)\vec{G}+(\vec{G}\cdot\nabla)\vec{F}+\vec{F}\times(\nabla\times\vec{G})+\vec{G}\times(\nabla\times\vec{F})$

$$\nabla(\vec{F}\cdot\vec{G})=\nabla_F(\vec{F}\cdot\vec{G})+\nabla_G(\vec{F}\cdot\vec{G})$$

關鍵在於如何求出 $\nabla_F(\vec{F}\cdot\vec{G})$ 與 $\nabla_G(\vec{F}\cdot\vec{G})$

由 $\vec{F}\times(\nabla\times\vec{G})=\vec{F}\times(\nabla_G\times\vec{G})=\nabla_G(\vec{F}\cdot\vec{G})-(\vec{F}\cdot\nabla_G)\vec{G}$

$$=\nabla_G(\vec{F}\cdot\vec{G})-(\vec{F}\cdot\nabla)\vec{G}$$

$$\Rightarrow\nabla_G(\vec{F}\cdot\vec{G})=\vec{F}\times(\nabla\times\vec{G})+(\vec{F}\cdot\nabla)\vec{G}$$

同理: $\vec{G}\times(\nabla\times\vec{F})=\nabla_F(\vec{F}\cdot\vec{G})-(\vec{G}\cdot\nabla)\vec{F}$

$$\Rightarrow\nabla_F(\vec{F}\cdot\vec{G})=\vec{G}\times(\nabla\times\vec{F})+(\vec{G}\cdot\nabla)\vec{F}$$

$\therefore\ \nabla(\vec{F}\cdot\vec{G})=\nabla_F(\vec{F}\cdot\vec{G})+\nabla_G(\vec{F}\cdot\vec{G})$

$$=(\vec{F}\cdot\nabla)\vec{G}+(\vec{G}\cdot\nabla)\vec{F}+\vec{F}\times(\nabla\times\vec{G})+\vec{G}\times(\nabla\times\vec{F})$$

(3) $\nabla\times(\nabla\times\vec{F})=\nabla(\nabla\cdot\vec{F})-\nabla^2\vec{F}$

$$\nabla\times\vec{F}=(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z})\vec{i}+(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x})\vec{j}+(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})\vec{k}$$

$$\nabla\times(\nabla\times\vec{F})=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ (\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}) & (\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}) & (\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})\end{vmatrix}$$

$\vec{i}$ 分量： $\frac{\partial}{\partial y}(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})-\frac{\partial}{\partial z}(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x})=\frac{\partial^2 Q}{\partial x\partial y}-\frac{\partial^2 P}{\partial y^2}-\frac{\partial^2 P}{\partial z^2}+\frac{\partial^2 R}{\partial x\partial z}$

$$=\frac{\partial^2 Q}{\partial x\partial y}-\frac{\partial^2 P}{\partial y^2}-\frac{\partial^2 P}{\partial z^2}+\frac{\partial^2 R}{\partial x\partial z}-\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial x\partial x}$$

$$=\frac{\partial}{\partial x}(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z})-\nabla^2 P$$

$$=\frac{\partial}{\partial x}(\nabla\cdot\vec{F})-\nabla^2 P$$

同理可得 $\vec{j}$ 分量： $\frac{\partial}{\partial y}(\nabla\cdot\vec{F})-\nabla^2 Q$

$\vec{k}$ 分量： $\frac{\partial}{\partial z}(\nabla\cdot\vec{F})-\nabla^2 R$

$$\therefore\nabla\times(\nabla\times\vec{F})=\frac{\partial}{\partial x}(\nabla\cdot\vec{F})\vec{i}-\nabla^2(P\vec{i})+\frac{\partial}{\partial y}(\nabla\cdot\vec{F})\vec{j}-\nabla^2(Q\vec{j})$$

$$+\frac{\partial}{\partial z}(\nabla\cdot\vec{F})\vec{k}-\nabla^2(R\vec{k})$$

$$=\nabla(\nabla\cdot\vec{F})-\nabla^2\vec{F}$$

2. $\vec{F}(x,y,z)=F_1\vec{i}+F_2\vec{j}+F_3\vec{k}$

由 $\nabla\cdot\vec{F}=2xy^3+8x^5yz-6y^4z^5\ \Rightarrow\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_2}{\partial z}=2xy^3+8x^5yz-6y^4z^5$

由 $\nabla\times\vec{F}=(-4y^3z^6-4x^5y^2)\,\vec{i}-4z^3\ \vec{j}+(20x^4y^2z-3x^2y^2)\,\vec{k}$

$$\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}=-4y^3z^6-4x^5y^2$$

$$\Rightarrow\ \frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}=-4z^3$$

$$\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}=20x^4y^2z-3x^2y^2$$

$\because$ $\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}=-4y^3z^6-4x^5y^2$ 存在 $4y^3z^6$ 項

$\therefore$ 由 $\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}=2xy^3+8x^5yz-6y^4z^5$ 可知 $\frac{\partial F_3}{\partial z}=-6y^4z^5$

$$\Rightarrow F_3=-y^4z^6+f_3(x,y)$$

由 $\frac{\partial F_1}{\partial z}-\frac{\partial F_3}{\partial x}=-4z^3$ 可得 $F_1=-z^4+\frac{\partial f_3(x,y)}{\partial x}z+f_1(x,y)$

由 $\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}=20x^4y^2z-3x^2y^2$ 可得

$$F_2=4x^5y^2z-x^3y^2+\frac{\partial f_3(x,y)}{\partial y}z+\int\frac{\partial f_1(x,y)}{\partial y}dx$$

又 $\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_2}{\partial z}=2xy^3+8x^5yz-6y^4z^5$

$$\frac{\partial F_1}{\partial x}=\frac{\partial^2 f_3(x,y)}{\partial x\partial x}z+\frac{\partial f_1(x,y)}{\partial x}$$

$$\frac{\partial F_2}{\partial y}=8x^5yz-2x^3y+\frac{\partial^2 f_3(x,y)}{\partial y\partial y}z+\int\frac{\partial^2 f_1(x,y)}{\partial y\partial y}dx$$

$$\frac{\partial F_3}{\partial z}=-6y^4z^5$$

比較後可假設 $f_3(x,y)=0$ 與 $\int\frac{\partial^2 f_1(x,y)}{\partial y\partial y}dx=2x^3y$

$$\Rightarrow f_1(x,y)=x^2y^3$$

$\therefore$ $F_1=-z^4+x^2y^3$，$F_2=4x^5y^2z$，$F_3=-y^4z^6$

3. $\vec{r}(t)=x\vec{i}+y\vec{j}=a\cos t\,\vec{i}+a\sin t\,\vec{j}$ $\Rightarrow x=a\cos t,\quad y=a\sin t$

$$ds=\sqrt{(x')^2+(y')^2}\,dt=a\,dt$$

$$\int_C x^2y\,ds=a^4\int_0^\pi\cos^2 t\cdot\sin t\,dt=-a^4\int_0^\pi\cos^2 t\cdot d(\cos t)=-\frac{a^4}{3}\cos^3 t\Big|_0^\pi=\frac{2}{3}a^4$$

4. 由 $3z=4y$ $\Rightarrow z=\frac{4}{3}y$ 代入 $x^2+y^2+z^2=1$

可得 $x^2+y^2+(\frac{4}{3}y)^2=1$ $\Rightarrow x^2+\frac{25}{9}y^2=1$

令 $x=\cos t$，$y=\frac{3}{5}\sin t$ 可得 $z=\frac{4}{5}\sin t$

$$ds = \sqrt{(x')^2 + (y')^2 + (z')^2}\ dt = dt$$

$$\int_C (xy+z)\,ds = \int_0^{2\pi} (\cos t \cdot \frac{3}{5}\sin t + \frac{4}{5}\sin t)\,dt = \int_0^{2\pi} (\frac{3}{10}\sin 2t + \frac{4}{5}\sin t)\,dt = 0$$

5. $y = 3x - x^2 \quad \Rightarrow dy = (3-2x)dx$

$$\int_C y^2 dx - xy\,dy = \int_0^3 (3x-x^2)^2 dx - x(3x-x^2)\cdot(3-2x)dx$$

$$= \int_0^3 (9x^2 - 6x^3 + x^4 - 9x^2 + 3x^3 + 6x^3 - 2x^4)\,dx$$

$$= \int_0^3 (3x^3 - x^4)\,dx$$

$$= \frac{3^5}{20}$$

6. 自 $(5,6)$ 至 $(3,3)$ 之直線 C_1 方程式為 $y = \frac{3}{2}x - \frac{3}{2} \quad \Rightarrow dy = \frac{3}{2}dx$

$$\therefore \int_{C_1} xy\,dx + (3x-y^2)\,dy = \int_5^3 x(\frac{3}{2}x - \frac{3}{2})dx + [3x - (\frac{3}{2}x - \frac{3}{2})^2]\cdot\frac{3}{2}dx = -10$$

路徑為折線之積分值為

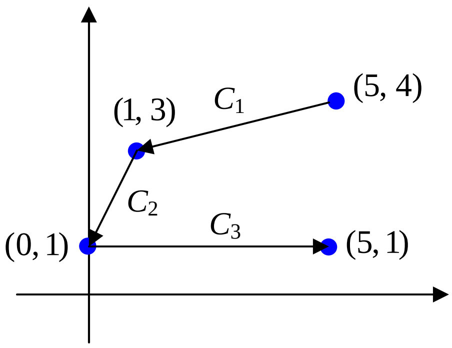
$$\int_{C_2+C_3} xy\,dx + (3x-y^2)\,dy = \int_{C_2} xy\,dx + (3x-y^2)\,dy + \int_{C_3} xy\,dx + (3x-y^2)\,dy$$

$$= \int_6^3 (15-y^2)dy + \int_5^3 3x\,dx = -6$$

$$\because \int_{C_1} xy\,dx + (3x-y^2)\,dy \neq \int_{C_2+C_3} xy\,dx + (3x-y^2)\,dy$$

$\therefore \vec{F}$ 不是保守場

7.



$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6x^2 & -2x & 0 \end{vmatrix} = -2\vec{k} \qquad \Rightarrow \text{ 此為非保守場}$$

又 $\vec{F} = 6x^2\,\vec{i} - 2x\,\vec{j}$ 且 $d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$ $\Rightarrow$ $\int_c \vec{F}\cdot d\vec{r} = \int_c 6x^2\,dx - 2x\,dy$

C_1: $(5, 4) \to (1, 3) \Rightarrow y = \frac{1}{4}(x-5)+4 \quad \Rightarrow dy = \frac{1}{4}dx$

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_1} 6x^2\,dx - 2x\,dy = \int_5^1 (6x^2 - \frac{x}{2})\,dx = -242$$

C_2: $(1, 3) \to (0, 1) \Rightarrow y = 2x+1 \quad \Rightarrow dy = 2dx$

$$\int_{C_2} \vec{F}\cdot d\vec{r} = \int_{C_2} 6x^2\,dx - 2x\,dy = \int_1^0 (6x^2 - 4x)\,dx = 0$$

C_3: $(0, 1) \to (5, 1) \Rightarrow y = 1 \quad \Rightarrow dy = 0dx$

$$\int_{C_3} \vec{F}\cdot d\vec{r} = \int_{C_3} 6x^2\,dx - 2x\,dy = \int_0^5 6x^2\,dx = 250$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_{C_1+C_2+C_3} \vec{F}\cdot d\vec{r} = 8$$