國立臺灣海洋大學河海工程學系 2026 工程數學(二)2AB 班課後練習 04 參考解答

系級：________________　　　學號：______________　　　姓名：_____________

1. 試求空間曲面 $z = x^2y^2 + y + 2$ 在位置 $(1, 0, 2)$ 之單位法向量與曲面 $z = x^3y^3 + x + 3$ 在位置 $(0, 0, 3)$ 之單位法向量。

2. 求曲面 $x^2y + z = 3$ 與 $x\ln z - y^2 = -4$ 在交點 $(-1, 2, 1)$ 之夾角?

3. 試求曲面 $z = \sin(xy)$ 在位置 $(1, 0, 0)$ 之切平面與法線方程式。

4. 已知 $f = xy^2 + 3x^2z$，$\vec{A} = y^2\vec{i} + (y^2 - x^2)\vec{j} + 2z^2\vec{k}$，試求:

   (1) $\nabla\cdot(\nabla f)$ (2) $\nabla\times(\nabla f)$ (3) $\nabla\cdot(\nabla\times\vec{A})$ (4) $\nabla\times(\nabla\times\vec{A})$

5. 已知某山脈高度分佈為 $h(x, y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)$，試問:

   (1) 山頂位置。

   (2) 山頂高度。

   (3) 位置 $(1, 1)$ 之最陡坡度與方向

   (4) 請計算 $\nabla\cdot\nabla h$ 與 $\nabla\times\nabla h$ 之值。

參考解答:

1. $z = x^2y^2 + y + 2$

$$\therefore\ \vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = x\vec{i} + y\vec{j} + (x^2y^2 + y + 2)\vec{k}$$

在位置 $(1, 0, 2)$

$$\vec{r}_x = x\vec{i} + (2xy^2)\vec{k} = \vec{i}$$

$$\vec{r}_y = \vec{j} + (2x^2y + 1)\vec{k} = \vec{j} + \vec{k}$$

$$\vec{n} = \frac{\vec{r}_x \times \vec{r}_y}{\left|\vec{r}_x \times \vec{r}_y\right|} = \frac{\vec{k} - \vec{j}}{\sqrt{2}}$$

$$z = x^3y^3 + x + 3$$

$$\therefore\ \vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = x\vec{i} + y\vec{j} + (x^3y^3 + x + 3)\vec{k}$$

在位置 $(0, 0, 3)$

$$\vec{r}_x = \vec{i} + (3x^2y^3 + 1)\vec{k} = \vec{i} + \vec{k}$$

$$\vec{r}_y = \vec{j} + (3x^3y^2)\vec{k} = \vec{j}$$

$$\vec{n} = \frac{\vec{r}_x \times \vec{r}_y}{\left|\vec{r}_x \times \vec{r}_y\right|} = \frac{\vec{k} - \vec{i}}{\sqrt{2}}$$

2. $x^2 y + z = 3 \quad \Rightarrow z = 3 - x^2 y$

$\therefore \quad \vec{r} = x\vec{i} + y\vec{j} + (3 - x^2 y)\vec{k}$

曲面 $x^2 y + z = 3$ 在位置 $(-1, 2, 1)$ 之單位法向量為

$\vec{r}_x = 1\vec{i} + 0\vec{j} - 2xy\vec{k} = \vec{i} + 4\vec{k}$

$\vec{r}_y = 0\vec{i} + 1\vec{j} - x^2\vec{k} = \vec{j} - \vec{k}$

$$\vec{n}_1 = \frac{\vec{r}_x \times \vec{r}_y}{|\vec{r}_x \times \vec{r}_y|} = \frac{-4\vec{i} + \vec{j} + \vec{k}}{\sqrt{18}}$$

$x \ln z - y^2 = -4 \quad \Rightarrow z = e^{\frac{y^2-4}{x}}$

$\therefore \quad \vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = x\vec{i} + y\vec{j} + e^{\frac{y^2-4}{x}}\vec{k}$

曲面 $x \ln z - y^2 = -4$ 在位置 $(-1, 2, 1)$ 之單位法向量為

$$\vec{r}_x = 1\vec{i} + 0\vec{j} - e^{\frac{y^2-4}{x}} \cdot \frac{y^2 - 4}{x^2}\vec{k} = \vec{i}$$

$$\vec{r}_y = 0\vec{i} + 1\vec{j} + e^{\frac{y^2-4}{x}} \cdot \frac{2y}{x}\vec{k} = \vec{j} - 4\vec{k}$$

$$\vec{n}_2 = \frac{\vec{r}_\theta \times \vec{r}_z}{|\vec{r}_\theta \times \vec{r}_z|} = \frac{4\vec{j} + \vec{k}}{\sqrt{17}}$$

$\therefore$兩曲面之夾角為 $\vec{n}_1 \cdot \vec{n}_2 = |\vec{n}_1||\vec{n}_2|\cos\theta$

$$\Rightarrow \theta = \cos^{-1}\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \cos^{-1}\frac{5}{\sqrt{18}\,\sqrt{17}} = 1.2809 \text{ (rad)} = 73.39^\circ$$

3. 由曲面 $z = \sin(xy)$ 可知 $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = x\vec{i} + y\vec{j} + \sin(xy)\vec{k}$

在位置 $(1, 0, 0)$

$\vec{r}_x = 1\vec{i} + 0\vec{j} + y\cos(xy)\vec{k} = \vec{i}$

$\vec{r}_y = 0\vec{i} + 1\vec{j} + x\cos(xy)\vec{k} = \vec{j} + \vec{k}$

$$\vec{n} = \frac{\vec{r}_x \times \vec{r}_y}{|\vec{r}_x \times \vec{r}_y|} = -\frac{1}{\sqrt{2}}\vec{j} + \frac{1}{\sqrt{2}}\vec{k}$$

切平面方程式為 $(\vec{r}-\vec{r}_0)\cdot\vec{n}=0 \quad \Rightarrow (x-1,\ y,\ z)\cdot(0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})=0$

$$\Rightarrow -\frac{1}{\sqrt{2}}y+\frac{1}{\sqrt{2}}z=0$$

$$\Rightarrow y-z=0$$

法線方程式為 $(\vec{r}-\vec{r}_0)\times\vec{n}=0$

$$\Rightarrow (x-1,\ y,\ z)\times(0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})=0$$

$$\Rightarrow \frac{1}{\sqrt{2}}(y+z)\vec{i}-\frac{1}{\sqrt{2}}(x-1)\,\vec{j}-\frac{1}{\sqrt{2}}(x-1)\,\vec{k}=0$$

$\Rightarrow x=1$ 且 $y+z=0$

或是 $(\vec{r}-\vec{r}_0)\times\vec{n}=0$ 即 $(\vec{r}-\vec{r}_0)//\vec{n}=0$

$$\Rightarrow \frac{x-1}{0}=\frac{y}{-\frac{1}{\sqrt{2}}}=\frac{z}{\frac{1}{\sqrt{2}}}=t$$

$$\Rightarrow x=1,\quad y=-\frac{1}{\sqrt{2}}t,\quad z=\frac{1}{\sqrt{2}}t$$

由 $y=-\frac{1}{\sqrt{2}}t$ 與 $z=\frac{1}{\sqrt{2}}t$ 可得 $y+z=2$

4. (1) $\nabla\cdot(\nabla f)=\nabla^2 f=\frac{\partial^2(xy^2+3x^2z)}{\partial x^2}+\frac{\partial^2(xy^2+3x^2z)}{\partial y^2}+\frac{\partial^2(xy^2+3x^2z)}{\partial z^2}$

$$=2x+6z$$

(2) $$\nabla\times(\nabla f)=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial(xy^2+3x^2z)}{\partial x} & \frac{\partial(xy^2+3x^2z)}{\partial y} & \frac{\partial(xy^2+3x^2z)}{\partial z}\end{vmatrix}$$

$$=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y^2+6xz & 2xy & 3x^2\end{vmatrix}=6x+6x+2y-2y=0$$

(3) $$\nabla\times\vec{A}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y^2 & y^2-x^2 & 2z^2\end{vmatrix}=-2(x+y)\vec{k}$$

$$\nabla\cdot(\nabla\times\vec{A})=-2\frac{\partial(x+y)}{\partial z}=0$$

(4) $\nabla\times(\nabla\times\vec{A})=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 0 & 0 & -2(x+y)\end{vmatrix}=-2\vec{i}+2\vec{j}$

5. (1) 最高點: $\frac{\partial h}{\partial x}=0 \quad \Rightarrow 10(2y-6x-18)=0$

$\frac{\partial h}{\partial y}=0 \quad \Rightarrow 10(2x-8y+28)=0$

解聯立可得 $x=-2$, $y=3$

(2) $h(-2,3)=10(-12-12-36+36+84+12)=720$

(3) $\nabla h=\frac{\partial h}{\partial x}\vec{i}+\frac{\partial h}{\partial y}\vec{j}=10(2y-6x-18)\,\vec{i}+10(2x-8y+28)\,\vec{j}$

在位置 $(1,1)$，$\nabla h=-220\,\vec{i}+220\,\vec{j}$

$\therefore$ 最陡方向為 $-\vec{i}+\vec{j}$，最陡坡度為 $|\nabla h|=220\sqrt{2}$

(4) $\nabla\cdot\nabla h=\frac{\partial(10(2y-6x-18))}{\partial x}+\frac{\partial(10(2x-8y+28))}{\partial y}=-140$

$\nabla\times\nabla h=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 10(2y-6x-18) & 10(2x-8y+28) & 0\end{vmatrix}=0$