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1. 試證明：

$$(1) \text{ 純量四重積 } (\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})$$

$$(2) \text{ 向量四重積 } (\vec{A} \times \vec{B}) \times (\vec{C} \times \vec{D}) = [\vec{A}\vec{B}\vec{D}]\vec{C} - [\vec{A}\vec{B}\vec{C}]\vec{D}$$

2. 對於三個已知向量 $\vec{A} = 3\hat{i} + 3\hat{j} - 2\hat{k}$ 、 $\vec{B} = -\hat{i} - 4\hat{j} + 2\hat{k}$ 與 $\vec{C} = 2\hat{i} + 2\hat{j} + \hat{k}$ ，

試求： $\vec{A} \cdot (\vec{B} \times \vec{C})$ ， $(\vec{A} \times \vec{B}) \times \vec{C}$ 與 $\vec{A} \times (\vec{B} \times \vec{C})$ 。

3. 試求橢圓 $\frac{1}{4}x^2 + y^2 = 1$ 在點 $P(\sqrt{2}, \frac{1}{\sqrt{2}})$ 的切線。

4. 試畫出下列各曲線圖形並計算其曲線長度

$$(1) \text{ 懸鍊線 } \vec{r}(t) = (t, \cosh t) \quad (t=0 \text{ 到 } t=2)$$

$$(2) \text{ 內擺線 } \vec{r}(t) = (a \cos^3 t, a \sin^3 t) \quad (t=0 \text{ 到 } t=2\pi, a=2)$$

$$(3) \text{ 心臟線 } \vec{r}(t) = (\rho \cos t, \rho \sin t) \quad \text{又 } \rho = a(1 - \cos t) \quad (t=0 \text{ 到 } t=2\pi, a=2)$$

$$(4) \text{ 腎形線 } \vec{r}(t) = (3a \cos t + a \cos 3t, 3a \sin t + a \sin 3t) \quad (t=0 \text{ 到 } t=2\pi, a=2)$$

5. 試求下列質點運動軌跡在 $t = \frac{5\pi}{6}$ 時之速度、切線加速度與法線加速度

$$(1) \vec{r}(t) = (\sin 2t, \cos t)$$

$$(2) \vec{r}(t) = (\sin t, \cos t, \cos 2t)$$

參考解答：

1. (1)

$$\begin{aligned} (\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) &= \vec{P} \cdot (\vec{C} \times \vec{D}) \\ &= \vec{C} \cdot (\vec{D} \times \vec{P}) \\ &= \vec{C} \cdot [\vec{D} \times (\vec{A} \times \vec{B})] \\ &= \vec{C} \cdot [(\vec{D} \cdot \vec{B})\vec{A} - (\vec{D} \cdot \vec{A})\vec{B}] \\ &= (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C}) \end{aligned}$$

(2)

$$\begin{aligned} (\vec{A} \times \vec{B}) \times (\vec{C} \times \vec{D}) &= \vec{P} \times (\vec{C} \times \vec{D}) \\ &= [(\vec{P} \cdot \vec{D})\vec{C} - (\vec{P} \cdot \vec{C})\vec{D}] \\ &= [((\vec{A} \times \vec{B}) \cdot \vec{D})\vec{C} - ((\vec{A} \times \vec{B}) \cdot \vec{C})\vec{D}] \\ &= [\vec{A}\vec{B}\vec{D}]\vec{C} - [\vec{A}\vec{B}\vec{C}]\vec{D} \end{aligned}$$

$$2. \vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} 3 & 3 & -2 \\ -1 & -4 & 2 \\ 2 & 2 & 1 \end{vmatrix} = -21$$

$$(\vec{A} \times \vec{B}) \times \vec{C} = -\vec{C} \times (\vec{A} \times \vec{B}) = -(\vec{C} \cdot \vec{B})\vec{A} + (\vec{C} \cdot \vec{A})\vec{B} = 8\vec{A} + 10\vec{B} = 14\hat{i} - 16\hat{j} + 4\hat{k}$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C} = 10\vec{B} + 19\vec{C} = 28\hat{i} - 2\hat{j} + 39\hat{k}$$

$$3. \vec{r}(t) = 2\cos t \vec{i} + \sin t \vec{j}$$

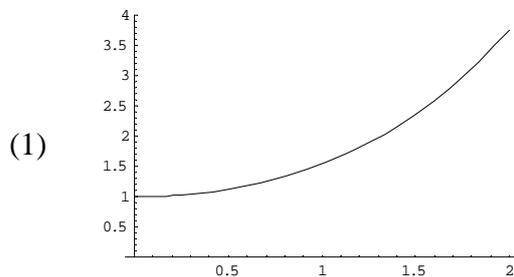
$$\vec{r}'(t) = -2\sin t \vec{i} + \cos t \vec{j}$$

$$P \text{ 點切向量 } \vec{r}'\left(\frac{\pi}{4}\right) = -\sqrt{2}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$$

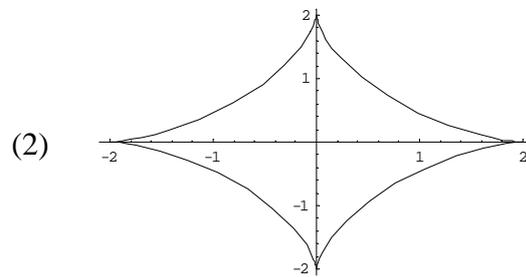
$$\text{切線參數方程式為 } \frac{x-\sqrt{2}}{-\sqrt{2}} = \frac{y-\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = t$$

$$\Rightarrow x = -\sqrt{2}t + \sqrt{2}, \quad y = \frac{1}{\sqrt{2}}t + \frac{1}{\sqrt{2}}$$

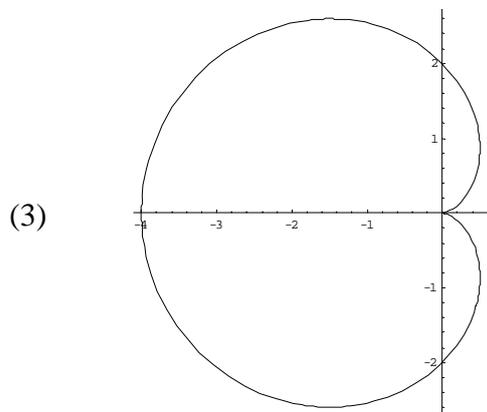
4.



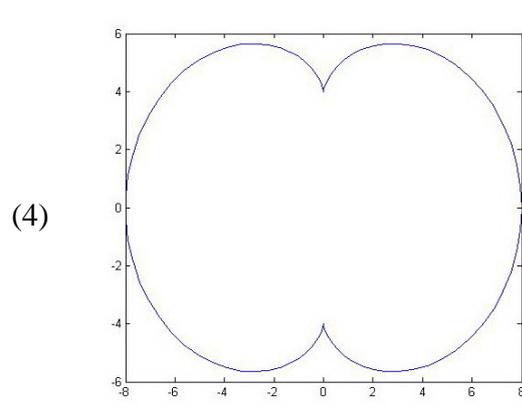
$$L = 3.627$$



$$L = 6a = 12$$



$$L = 8a = 16$$



$$L = 24a = 48$$

$$5. (1) \vec{v} = 2\cos 2t \vec{i} - \sin t \vec{j} = \vec{i} - \frac{1}{2} \vec{j}$$

$$\vec{a} = -4\sin 2t \vec{i} - \cos t \vec{j} = 2\sqrt{3} \vec{i} + \frac{\sqrt{3}}{2} \vec{j}$$

$$\vec{a}_t = \frac{8\sin 4t - \sin 2t}{-4\cos 4t + \cos 2t - 5} (2\cos 2t \vec{i} - \sin t \vec{j}) = \frac{7\sqrt{3}}{5} \vec{i} - \frac{7\sqrt{3}}{10} \vec{j}$$

$$\vec{a}_n = \vec{a} - \vec{a}_t = \frac{3\sqrt{3}}{5} \vec{i} + \frac{6\sqrt{3}}{5} \vec{j}$$

$$(2) \vec{v} = \cos t \vec{i} - \sin t \vec{j} - 2\sin 2t \vec{k} = -\frac{\sqrt{3}}{2} \vec{i} - \frac{1}{2} \vec{j} + \sqrt{3} \vec{k}$$

$$\vec{a} = -\sin t \vec{i} - \cos t \vec{j} - 4\cos 2t \vec{k} = -\frac{1}{2} \vec{i} + \frac{\sqrt{3}}{2} \vec{j} - 2\vec{k}$$

$$\vec{a}_t = \frac{4\sin 4t}{-3 + 2\cos 4t} (\cos t \vec{i} - \sin t \vec{j} - 2\sin 2t \vec{k}) = -\frac{3}{4} \vec{i} - \frac{\sqrt{3}}{4} \vec{j} + \frac{3}{2} \vec{k}$$

$$\vec{a}_n = \vec{a} - \vec{a}_t = \frac{1}{4} \vec{i} - \frac{3\sqrt{3}}{4} \vec{j} - \frac{7}{2} \vec{k}$$