

系級：_____ 學號：_____ 姓名：_____

1. 試判定下列函數那些是奇函數，那些是偶函數，那些是非奇非偶函數？(12%)

(請說明判斷理由，無說明者不給分)

(1) $f(x) = \sin^3 x + 1$ (2) $f(x) = x \sin^3 x + 1$

(3) $f(x) = \cos^3 x + 1$ (4) $f(x) = x \cos^3 x + 1$

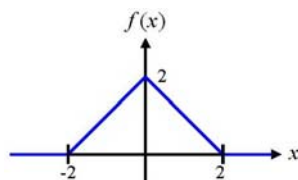
2. 試求函數 $f(x) = \cos^3 2x + \sin^5 x$ 的傅立葉級數展開。(8%)

3. 給一週期函數 $f(x) = x^2$, $-1 < x < 1$ 且 $f(x) = f(x+2)$, 試求:

(1) 此函數之傅立葉級數展開。(6%) (2) $\sum_{n=1}^{\infty} \frac{1}{n^2} = ?$ (4%)

(3) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = ?$ (4%) (4) $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = ?$ (4%) (5) $\sum_{n=1}^{\infty} \frac{1}{n^4} = ?$ (4%)

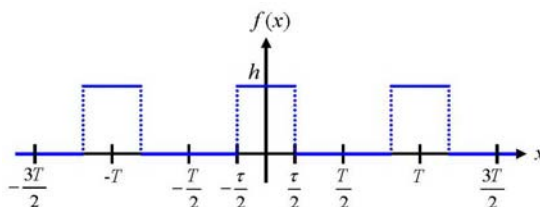
4. 給一函數如下圖所示



(1) 試求 $f(x)$ 的傅立葉積分表示式。(7%)

(2) 試問 $\int_0^{\infty} \frac{1 - \cos 2\omega}{\omega^2} d\omega = ?$ (3%)

5. 把寬為 τ , 高為 h , 週期為 T 的矩形波(如下圖)展開成複數形式傅立葉級數。(10%)



6. 若 $f(t) = \begin{cases} 1, & 0 \leq t \leq \pi \\ 0, & t < 0 \text{ and } t > \pi \end{cases}$ 且 $g(t) = \begin{cases} \sin t, & 0 \leq t \leq \pi \\ 0, & t < 0 \text{ and } t > \pi \end{cases}$

(1) 試問其傅立葉轉換 $F(\omega)$ 與 $G(\omega)$ 為何？(10%)

(2) 若 $p(t) = g(t - \pi)$, 則其傅立葉轉換 $P(\omega) = ?$ (4%)

(3) 若 $q(t) = \frac{dg(t)}{dt}$, 則其傅立葉轉換 $Q(\omega) = ?$ (4%)

(4) $h(t) = f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau = ?$ (5%)

(hint: 請將 t 分成 $t < 0$, $0 \leq t \leq \pi$, $\pi \leq t \leq 2\pi$ 與 $t > 2\pi$ 討論)

(5) $H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-i\omega t} dt = ?$ (5%)

7. 已知函數 $f(x) = e^{-a|x|}$ 且 $a > 0$ 的傅立葉轉換為 $F(\omega) = \mathcal{F}[f(x)] = \frac{2a}{\omega^2 + a^2}$ ，試求

函數 $\frac{6e^{i\omega} \cos(4\omega)}{9 + \omega^2}$ 的傅立葉逆轉換為何? (10%)

傅立葉級數展開

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2n\pi x}{T} + b_n \sin \frac{2n\pi x}{T} \right)$$

$$a_0 = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) dx, \quad a_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) \cos \frac{2n\pi x}{T} dx, \quad b_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) \sin \frac{2n\pi x}{T} dx$$

傅立葉級數之 Parseval 恆等式： $\frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f^2(x) dx = a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$

傅立葉積分： $f(x) = \int_0^{\infty} [A(\omega) \cos \omega x + B(\omega) \sin \omega x] d\omega$

$$\text{其中 } A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos \omega x dx, \quad B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin \omega x dx$$

傅立葉複數形式級數展開

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n x}, \quad \text{其中 } \omega_n = \frac{2n\pi}{T}, \quad c_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) e^{-i\omega_n x} dx$$

傅立葉轉換： $F(\omega) = \mathcal{F}[f(x)] = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$

傅立葉反轉換： $f(x) = \mathcal{F}^{-1}[F(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x} d\omega$

傅立葉轉換的 Parseval 恆等式： $\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$

Convolution： $f * g = \int_{-\infty}^{\infty} f(t - \tau) g(\tau) d\tau \Rightarrow \mathcal{F}[f(t) * g(t)] = F(\omega) \cdot G(\omega)$

$\mathcal{F}[f'(t)] = i\omega F(\omega) \Rightarrow \mathcal{F}[f^{(n)}(t)] = (i\omega)^n F(\omega)$

$\mathcal{F}[t^n f(t)] = i^n \frac{d^n}{d\omega^n} F(\omega)$

$\int_a^b f(x) \delta(x - x_0) dx = f(x_0) \quad \text{其中 } a < x_0 < b$

尤拉公式： $e^{\pm ix} = \cos x \pm i \sin x \Rightarrow \cos x = \frac{1}{2}(e^{ix} + e^{-ix})$ 與 $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$

Scaling： $\mathcal{F}[f(at)] = \frac{1}{a} F\left(\frac{\omega}{a}\right)$

Time shifting： $\mathcal{F}[f(t - T)] = e^{-i\omega T} F(\omega)$

Frequency shifting： $\mathcal{F}[e^{i\omega_0 t} f(t)] = F(\omega - \omega_0)$

參考解答：

1. 試判定下列函數那些是奇函數，那些是偶函數，那些是非奇非偶函數？(12%)

(請說明判斷理由，無說明者不給分)

$$(1) f(x) = \sin^3 x + 1 \quad (2) f(x) = x \sin^3 x + 1$$

$$(3) f(x) = \cos^3 x + 1 \quad (4) f(x) = x \cos^3 x + 1$$

偶函數： $f(x) = f(-x)$ ， 奇函數： $f(x) = -f(-x)$

$$(1) f(-x) = \sin^3(-x) + 1 = -\sin^3 x + 1$$

$\therefore$ 等號右式不等於 $f(x)$ ，也不等於 $-f(x)$

$\therefore$ 此為非奇非偶函數

$$(2) f(-x) = (-x) \sin^3(-x) + 1 = x \sin^3 x + 1 = f(x)$$

$\therefore$ 此為偶函數

$$(3) f(-x) = \cos^3(-x) + 1 = \cos^3 x + 1 = f(x)$$

$\therefore$ 此為偶函數

$$(4) f(-x) = (-x) \cos^3(-x) + 1 = -x \cos^3 x + 1$$

$\therefore$ 等號右式不等於 $f(x)$ ，也不等於 $-f(x)$

$\therefore$ 此為非奇非偶函數

2. 試求函數 $f(x) = \cos^3 2x + \sin^5 x$ 的傅立葉級數展開。

由尤拉公式可知：

$$\begin{aligned} \cos 2x &= \frac{e^{i2x} + e^{-i2x}}{2} \Rightarrow \cos^3 2x = \left(\frac{e^{i2x} + e^{-i2x}}{2} \right)^3 \\ &= \frac{1}{8} (e^{i6x} + 3e^{i2x} + 3e^{-i2x} + e^{-i6x}) \\ &= \frac{1}{4} \cos 6x + \frac{3}{4} \cos 2x \end{aligned}$$

$$\begin{aligned} \sin x &= \frac{e^{ix} - e^{-ix}}{2i} \Rightarrow \sin^5 x = \left(\frac{e^{ix} - e^{-ix}}{2i} \right)^5 \\ &= \frac{1}{32i} (e^{i5x} - 5e^{i3x} + 10e^{ix} - 10e^{-ix} + 5e^{-i3x} - e^{-i5x}) \\ &= \frac{1}{16} \sin 5x - \frac{5}{16} \sin 3x + \frac{5}{8} \sin x \end{aligned}$$

$$\therefore f(x) = \cos^3 2x + \sin^5 x = \frac{1}{4} \cos 6x + \frac{3}{4} \cos 2x + \frac{1}{16} \sin 5x - \frac{5}{16} \sin 3x + \frac{5}{8} \sin x$$

3. 給一週期函數 $f(x) = x^2$, $-1 < x < 1$ 且 $f(x) = f(x+2)$

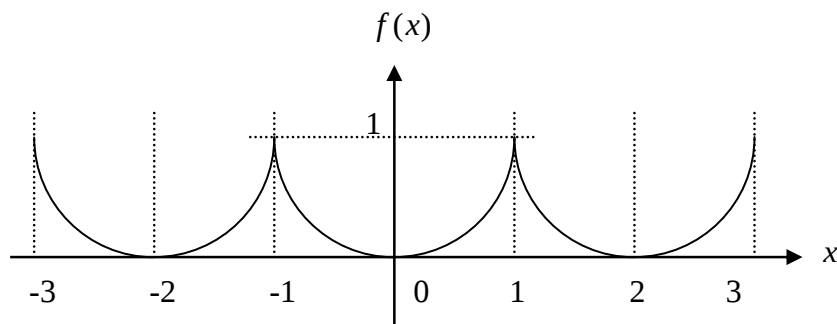
(1) 試求其傅立葉級數展開。(6%)

(2) $\sum_{n=1}^{\infty} \frac{1}{n^2} = ?$ (4%)

(3) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = ?$ (4%)

(4) $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = ?$ (4%)

(5) $\sum_{n=1}^{\infty} \frac{1}{n^4} = ?$ (4%)



(1) $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{2n\pi x}{T} + \sum_{n=1}^{\infty} b_n \sin \frac{2n\pi x}{T}$ (已知 $T = 2$, $b_n = 0$)

$$\Rightarrow f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x$$

$$a_0 = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) dx = \frac{1}{2} \int_{-1}^1 x^2 dx = \frac{1}{3}$$

$$a_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) \cos n\pi x dx = \int_{-1}^1 x^2 \cos n\pi x dx = \frac{4(-1)^n}{n^2 \pi^2}$$

$$= \left(\frac{x^2}{n\pi} \sin n\pi x + \frac{2x}{n^2 \pi^2} \cos n\pi x - \frac{2}{n^3 \pi^3} \sin n\pi x \right) \Big|_{-1}^1 = \frac{4(-1)^n}{n^2 \pi^2}$$

$$\therefore f(x) = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi x$$

(2) $f(1) = 1^2 = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

(3) $f(0) = 0^2 = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$

(4) $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \frac{\pi^2}{6} \dots\dots (1)$

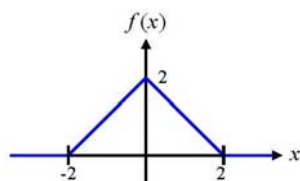
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12} \quad \text{..... (2)}$$

$$\begin{aligned} (1)+(2) \text{ 可得 } 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) &= \frac{\pi^2}{4} \\ \Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots &= \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8} \end{aligned}$$

$$(5) \text{ 應用 Parseval 定理: } \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f^2(x) dx = a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

$$\text{可得 } \frac{1}{2} \int_{-1}^1 x^4 dx = \frac{1}{9} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{16}{n^4 \pi^4} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

4. 給一函數如下圖所示



(1) 試求 $f(x)$ 的傅立葉積分表示式。(7%)

(2) 試問: $\int_0^{\infty} \frac{1 - \cos 2\omega}{\omega^2} d\omega = ?$ (3%)

(1) 由圖可知此為偶函數

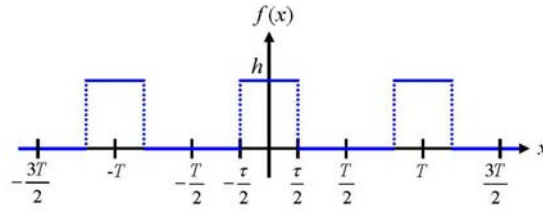
$$\therefore f(x) = \int_0^{\infty} A(\omega) \cos \omega x d\omega \quad (B(\omega) = 0)$$

$$\begin{aligned} A(\omega) &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos \omega x dx = \frac{2}{\pi} \int_0^2 (2-x) \cos \omega x dx \\ &= \frac{2}{\pi \omega^2} (1 - \cos 2\omega) \end{aligned}$$

$$\therefore f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{(1 - \cos 2\omega) \cos \omega x}{\omega^2} d\omega$$

$$(2) \quad f(0) = 2 = \frac{2}{\pi} \int_0^{\infty} \frac{1 - \cos 2\omega}{\omega^2} d\omega \Rightarrow \int_0^{\infty} \frac{1 - \cos 2\omega}{\omega^2} d\omega = \pi$$

5. 把寬為 τ ，高為 h ，週期為 T 的矩形波(如下圖)展開成複數形式傅立葉級數。
(10%)



由圖可知在一個週期 $[-T/2, T/2]$ 內，函數的表示為 $f(x) = \begin{cases} 0, & -\frac{T}{2} \leq x < -\frac{\tau}{2} \\ h, & -\frac{\tau}{2} \leq x < \frac{\tau}{2} \\ 0, & \frac{\tau}{2} \leq x < \frac{T}{2} \end{cases}$

函數 $f(x)$ 的複數形式傅立葉級數為 $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n x}$ 且 $\omega_n = \frac{2n\pi}{T}$

$$\therefore c_0 = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) dx = \frac{1}{T} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} h dx = \frac{h\tau}{T}$$

$$\begin{aligned} c_n &= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) e^{-i\omega_n x} dx = \frac{1}{T} \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} h e^{-i\frac{2n\pi}{T}x} dx = \frac{h}{T} \left(-\frac{T}{2n\pi i} e^{-i\frac{2n\pi}{T}x} \right) \bigg|_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \\ &= -\frac{h}{2n\pi i} (e^{-i\frac{n\pi\tau}{T}} - e^{i\frac{n\pi\tau}{T}}) \\ &= \frac{h}{n\pi} \sin \frac{n\pi\tau}{T} \quad (n = \pm 1, \pm 2, \pm 3, \dots) \end{aligned}$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n x} = \frac{h\tau}{T} + \frac{h}{\pi} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{n} \sin \frac{n\pi\tau}{T} e^{i\frac{2n\pi}{T}x}$$

6. 若 $f(t) = \begin{cases} 1, 0 \leq t \leq \pi \\ 0, t < 0 \text{ and } t > \pi \end{cases}$ 且 $g(t) = \begin{cases} \sin t, 0 \leq t \leq \pi \\ 0, t < 0 \text{ and } t > \pi \end{cases}$

(1) 試問其傅立葉轉換 $F(\omega)$ 與 $G(\omega)$ 為何? (8%)

(2) 若 $p(t) = g(t - \pi)$, 則其傅立葉轉換 $P(\omega) = ?$ (4%)

(3) 若 $q(t) = \frac{dg(t)}{dt}$, 則其傅立葉轉換 $Q(\omega) = ?$ (4%)

(4) $h(t) = f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t - \tau)d\tau = ?$ (5%)

(hint: 請將 t 分成 $t < 0$, $0 \leq t \leq \pi$, $\pi \leq t \leq 2\pi$ 與 $t > 2\pi$ 討論)

(5) $H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-i\omega t}dt = ?$ (5%)

$$(1) F(\omega) = \int_0^{\pi} 1 \cdot e^{-i\omega t} dt = -\frac{1}{i\omega} e^{-i\omega t} \Big|_0^{\pi} = \frac{i}{\omega} (e^{-i\omega\pi} - 1)$$

$$\begin{aligned} G(\omega) &= \int_0^{\pi} \sin t \cdot e^{-i\omega t} dt \\ &= -\frac{1}{i\omega} \sin t \cdot e^{-i\omega t} \Big|_0^{\pi} - \frac{1}{i^2 \omega^2} \cos t \cdot e^{-i\omega t} \Big|_0^{\pi} - \frac{1}{i^2 \omega^2} \int_0^{\pi} \sin t \cdot e^{-i\omega t} dt \\ \Rightarrow (1 - \frac{1}{\omega^2}) \int_0^{\pi} \sin t \cdot e^{-i\omega t} dt &= -\frac{1}{\omega^2} (e^{-i\omega\pi} + 1) \end{aligned}$$

$$\Rightarrow G(\omega) = \int_0^{\pi} \sin t \cdot e^{-i\omega t} dt = \frac{-\frac{1}{\omega^2} (e^{-i\omega\pi} + 1)}{(1 - \frac{1}{\omega^2})} = \frac{e^{-i\omega\pi} + 1}{1 - \omega^2}$$

(2) 由 time-shifting 定理:

$$P(\omega) = \mathcal{F}[p(t)] = \mathcal{F}[g(t - \pi)] = e^{-i\omega\pi} G(\omega) = \frac{e^{-2i\omega\pi} + e^{-i\omega\pi}}{1 - \omega^2}$$

或者由直接作傅立葉轉換亦可得

$$\begin{aligned} P(\omega) &= \int_{\pi}^{2\pi} \sin(t - \pi) e^{-i\omega t} dt \\ &= \frac{i}{\omega} \sin(t - \pi) \cdot e^{-i\omega t} \Big|_{\pi}^{2\pi} + \frac{1}{\omega^2} \cos(t - \pi) \cdot e^{-i\omega t} \Big|_{\pi}^{2\pi} \\ &\quad + \frac{1}{\omega^2} \int_{\pi}^{2\pi} \sin(t - \pi) \cdot e^{-i\omega t} dt \end{aligned}$$

$$\Rightarrow P(\omega) = \int_{\pi}^{2\pi} \sin(t - \pi) e^{-i\omega t} dt = \frac{e^{-2i\omega\pi} + e^{-i\omega\pi}}{1 - \omega^2}$$

(3) 由傅立葉微分公式可知:

$$Q(\omega) = \mathcal{F}[g'(t)] = i\omega G(\omega) = \frac{i\omega(e^{-i\omega\pi} + 1)}{1 - \omega^2}$$

$$(4) \quad h(t) = f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$$

$$(i) \quad t < 0 \quad \Rightarrow h(t) = 0$$

$$(ii) \quad 0 \leq t \leq \pi \quad \Rightarrow h(t) = \int_0^t \sin(t-\tau)d\tau = 1 - \cos t$$

$$(iii) \quad \pi \leq t \leq 2\pi \quad \Rightarrow h(t) = \int_{t-\pi}^{\pi} \sin(t-\tau)d\tau = 1 - \cos t$$

$$(iv) \quad t > 2\pi \quad \Rightarrow h(t) = 0$$

$$(5) \quad H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-i\omega t} dt = \mathcal{F}[f(t) * g(t)] = F(\omega)G(\omega)$$

$$= \frac{i}{\omega}(e^{-i\omega\pi} - 1) \cdot \frac{e^{-i\omega\pi} + 1}{1 - \omega^2} = \frac{i(e^{-2i\omega\pi} - 1)}{\omega - \omega^3} = \frac{-i(e^{-2i\omega\pi} - 1)}{\omega^3 - \omega}$$

$$\text{或者亦可由 } H(\omega) = \mathcal{F}[h(t)] = \int_{-\infty}^{\infty} h(t)e^{-i\omega t} dt$$

$$\begin{aligned} &= \int_0^{2\pi} (1 - \cos t)e^{-i\omega t} dt \\ &= \frac{1}{-i\omega} e^{-i\omega t} \Big|_0^{2\pi} - \frac{\frac{1}{-i\omega} \cos t \cdot e^{-i\omega t} \Big|_0^{2\pi} + \frac{1}{i^2 \omega^2} \sin t \cdot e^{-i\omega t} \Big|_0^{2\pi}}{1 - \frac{1}{\omega^2}} \\ &= \frac{i}{\omega}(e^{-i2\pi\omega} - 1) - \frac{i\omega(e^{-i2\omega\pi} - 1)}{\omega^2 - 1} \\ &= \frac{i(\omega^2 - 1)(e^{-i2\pi\omega} - 1) - i\omega^2(e^{-i2\omega\pi} - 1)}{\omega^3 - \omega} \\ &= \frac{-i(e^{-i2\pi\omega} - 1)}{\omega^3 - \omega} \end{aligned}$$

7. 已知函數 $f(x) = e^{-a|x|}$ 且 $a > 0$ 的傅立葉轉換為 $F(\omega) = \mathcal{F}[f(x)] = \frac{2a}{\omega^2 + a^2}$ ，試求

函數 $G(\omega) = \frac{6e^{i\omega} \cos(4\omega)}{9 + \omega^2}$ 的傅立葉逆轉換為何? (10%)

$$G(\omega) = \frac{6e^{i\omega} \cos(4\omega)}{9 + \omega^2} = \frac{6e^{i\omega}}{9 + \omega^2} \cdot \frac{e^{i4\omega} + e^{-i4\omega}}{2} = \frac{3}{9 + \omega^2} e^{i5\omega} + \frac{3}{9 + \omega^2} e^{-i3\omega}$$

$$\text{已知 } \mathcal{F}[e^{-a|x|}] = \frac{2a}{\omega^2 + a^2} \quad \Rightarrow f(x) = \mathcal{F}^{-1}\left[\frac{2a}{\omega^2 + a^2}\right] = e^{-a|x|}$$

$$H(\omega) = \frac{3}{9 + \omega^2} \quad \Rightarrow h(x) = \mathcal{F}^{-1}\left[\frac{3}{9 + \omega^2}\right] = \frac{1}{2} e^{-3|x|}$$

$$\mathcal{F}[f(x-T)] = e^{-i\omega T} F(\omega) \quad \Rightarrow \mathcal{F}^{-1}[e^{-i\omega T} F(\omega)] = f(x-T)$$

$$\therefore g(x) = \mathcal{F}^{-1}[G(\omega)] = h(x+5) + h(x-3) = \frac{1}{2}(e^{-3|x+5|} + e^{-3|x-3|})$$