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$$1. \mathbf{A} = \begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix}$$

(1) 試求 $\mathbf{A}$ 之特徵值、特徵向量並將 $\mathbf{A}$ 對角化。(2) 若 $f(x) = x^2 + 2x + 1$ ，試求 $f(\mathbf{A})$ 之特徵值與特徵向量。

$$2. \text{ 已知 } \mathbf{A}^{\frac{1}{2}} = \begin{bmatrix} -1 & 6 \\ -2 & 6 \end{bmatrix}$$

試求： $\mathbf{A}$ 、 $\det(\mathbf{A})$ 、 $\mathbf{A}^{-1}$ 、 $\mathbf{A}^{10}$ 、 $\mathbf{A}^3 - 2\mathbf{A}^2 - 3\mathbf{A} - 2\mathbf{I}$ 、 $e^{\mathbf{A}}$ 、 $\cos(\mathbf{A})$ **參考解答：**

$$1. (1) |\mathbf{A} - \lambda \mathbf{I}| = 0 \Rightarrow \begin{vmatrix} 11-\lambda & -4 & -7 \\ 7 & -2-\lambda & -5 \\ 10 & -4 & -6-\lambda \end{vmatrix} = 0 \Rightarrow \lambda(\lambda-1)(\lambda-2) = 0$$

$$\Rightarrow \lambda = 0, 1, 2$$

$$\text{當 } \lambda_1 = 0 \text{ 時，可得 } \mathbf{x}^1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{當 } \lambda_2 = 1 \text{ 時，可得 } \mathbf{x}^2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$\text{當 } \lambda_3 = 2 \text{ 時，可得 } \mathbf{x}^3 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 1 & 2 & 2 \end{bmatrix} \Rightarrow \mathbf{S}^{-1} = \begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 1 \\ 3 & -1 & -2 \end{bmatrix}$$

$$\mathbf{AS} = \mathbf{SD} \Rightarrow \mathbf{D} = \mathbf{S}^{-1}\mathbf{AS}$$

$$\Rightarrow \mathbf{D} = \begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 1 \\ 3 & -1 & -2 \end{bmatrix} \begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(2) $f(\mathbf{A})$ 之特徵值為 $f(\lambda) = \lambda^2 + 2\lambda + 1$

$$\therefore \lambda_1 = f(0) = 0, \lambda_2 = f(1) = 4, \lambda_3 = f(2) = 9$$

$$\text{特徵向量 } \mathbf{x}^1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{x}^2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \mathbf{x}^3 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$$2. \mathbf{B} = \mathbf{A}^{\frac{1}{2}} = \begin{bmatrix} -1 & 6 \\ -2 & 6 \end{bmatrix} \Rightarrow \mathbf{A} = \mathbf{B}^2 = \begin{bmatrix} -11 & 30 \\ -10 & 24 \end{bmatrix}$$

$$\det(\mathbf{A}) = 36$$

$$\mathbf{A}^{-1} = \frac{1}{36} \begin{bmatrix} 24 & -30 \\ 10 & -11 \end{bmatrix}$$

$$|\mathbf{A} - \lambda \mathbf{I}| = 0 \Rightarrow \begin{vmatrix} -11-\lambda & 30 \\ -10 & 24-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda-4)(\lambda-9) = 0 \Rightarrow \lambda = 4, 9$$

$$\text{當 } \lambda_1 = 4 \text{ 時, 可得 } \mathbf{x}^1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\text{當 } \lambda_2 = 9 \text{ 時, 可得 } \mathbf{x}^2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}, \mathbf{S}^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$$

$$\mathbf{AS} = \mathbf{SD} \Rightarrow \mathbf{A} = \mathbf{SDS}^{-1}$$

$$\mathbf{A}^{10} = \mathbf{SD}^{10}\mathbf{S}^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 4^{10} & 0 \\ 0 & 9^{10} \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 4 \cdot 4^{10} - 3 \cdot 9^{10} & -6 \cdot 4^{10} + 6 \cdot 9^{10} \\ 2 \cdot 4^{10} - 2 \cdot 9^{10} & -3 \cdot 4^{10} + 4 \cdot 9^{10} \end{bmatrix}$$

$$\begin{aligned} \mathbf{A}^3 - 2\mathbf{A}^2 - 3\mathbf{A} - 2\mathbf{I} &= \mathbf{S}(\mathbf{D}^3 - 2\mathbf{D}^2 - 3\mathbf{D} - 2\mathbf{I})\mathbf{S}^{-1} \\ &= \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 18 & 0 \\ 0 & 538 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1542 & 3120 \\ -1040 & 2098 \end{bmatrix} \end{aligned}$$

$$e^{\mathbf{A}} = \mathbf{S}e^{\mathbf{D}}\mathbf{S}^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} e^4 & 0 \\ 0 & e^9 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 4 \cdot e^4 - 3 \cdot e^9 & -6 \cdot e^4 + 6 \cdot e^9 \\ 2 \cdot e^4 - 2 \cdot e^9 & -3 \cdot e^4 + 4 \cdot e^9 \end{bmatrix}$$

$$\begin{aligned} \sin(\mathbf{A}) &= \mathbf{S} \cdot \sin(\mathbf{D}) \cdot \mathbf{S}^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \sin(4) & 0 \\ 0 & \sin(9) \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 4 \cdot \sin(4) - 3 \cdot \sin(9) & -6 \cdot \sin(4) + 6 \cdot \sin(9) \\ 2 \cdot \sin(4) - 2 \cdot e^9 & -3 \cdot \sin(4) + 4 \cdot \sin(9) \end{bmatrix} \end{aligned}$$