空間曲線補充講義

空間曲線

	時間參數 ***t*** 表示	空間參數 ***s*** 表示
位置向量	$\vec{r}(t)=x(t)\hat{i}+y(t)\hat{j}+z(t)\hat{k}$	$\vec{R}(s)=x(s)\hat{i}+y(s)\hat{j}+z(s)\hat{k}$
單位切向量	$\vec{t}(t)=\dfrac{\vec{r}\,'(t)}{\left\|\vec{r}\,'(t)\right\|}$	$\vec{T}(s)=\vec{R}'(s)$
單位法向量	$\vec{n}(t)=\dfrac{\vec{t}\,'(t)}{\left\|\vec{t}\,'(t)\right\|}$	$\vec{N}(s)=\dfrac{\vec{T}'(s)}{\left\|\vec{T}'(s)\right\|}=\dfrac{\vec{R}''(s)}{\left\|\vec{R}''(s)\right\|}=\dfrac{\vec{R}''(s)}{K(s)}$
曲率	$\kappa(t)=\dfrac{\left\|\vec{t}\,'(t)\right\|}{\left\|\vec{r}\,'(t)\right\|}=\dfrac{\left\|\vec{r}\,'\times\vec{r}\,''\right\|}{(\vec{r}\,'\cdot\vec{r}\,')^{\frac{3}{2}}}$	$K(s)=\left\|\vec{T}'(s)\right\|=\left\|\vec{R}''(s)\right\|$
單位 副法向量	$\vec{b}(t)=\vec{t}(t)\times\vec{n}(t)$	$\vec{B}(s)=\vec{T}(s)\times\vec{N}(s)$
扭率	$\tau=\dfrac{[\vec{r}\,'\ \vec{r}\,''\ \vec{r}\,''']}{\left\|\vec{r}\,'\times\vec{r}\,''\right\|^2}$	$\tau=\left\|\dfrac{d\vec{B}(s)}{ds}\right\|$

定義:

曲率: 以弧長參數 s 表示，單位切向量變化率的大小。 $\left(\kappa=\left|\dfrac{d\vec{T}(s)}{ds}\right|\right)$

扭率: $\tau=\left|\dfrac{d\vec{B}(s)}{ds}\right|$

路徑長計算(弧長):

微小路徑長度(弧長): $ds=\left|d\vec{r}(t)\right|=\left|dx\,\hat{i}+dy\,\hat{j}+dz\,\hat{k}\right|=\sqrt{(dx)^2+(dy)^2+(dz)^2}$

$$=\sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2+(\frac{dz}{dt})^2}\ dt$$

$$=\sqrt{[x'(t)]^2+[y'(t)]^2+[z'(t)]^2}\ dt$$

$$= \sqrt{|\vec{r}\,'(t)|^2}\ dt \quad = |\vec{r}\,'(t)|\ dt$$

時間 $t = t_1$ 到 $t = t_2$ 路徑長(弧長): $S = \int_C ds = \int_{t_1}^{t_2} |\vec{r}\,'(t)|\, dt$

$$= \int_{t_1}^{t_2} \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\ dt$$

到某個時刻 t 之路徑長(弧長):

$$s(t) = \int_{\tau=t_1}^{\tau=t} \sqrt{[x'(\tau)]^2 + [y'(\tau)]^2 + [z'(\tau)]^2}\ d\tau = \int_{\tau=t_1}^{\tau=t} |\vec{r}\,'(\tau)|\, d\tau$$

轉換關係:

單位切向量:

$$\vec{t}\,(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{\dfrac{d\vec{r}\,(t)}{dt}}{\left|\dfrac{d\vec{r}\,(t)}{dt}\right|} = \frac{\dfrac{d\vec{r}\,(t)}{dt}}{\dfrac{ds}{dt}} = \frac{d\vec{r}\,(t)}{ds} = \frac{d\vec{r}\,(t(s))}{ds} = \frac{d\vec{R}(s)}{ds} = \vec{R}'(s) = \vec{T}(s)$$

單位法向量:

$$\vec{n}(t) = \frac{\vec{t}\,'(t)}{|\vec{t}\,'(t)|} = \frac{\dfrac{ds}{dt}\dfrac{d\vec{t}\,(t)}{ds}}{\dfrac{ds}{dt}\left|\dfrac{d\vec{t}\,(t)}{ds}\right|} = \frac{\dfrac{d\vec{T}(s)}{ds}}{\left|\dfrac{d\vec{T}(s)}{ds}\right|} = \frac{\vec{T}'(s)}{|\vec{T}'(s)|} = \vec{N}(s)$$

曲率:

$$K(s) = |\vec{T}'(s)| = |\vec{R}''(s)| = \left|\frac{d\vec{T}(s)}{ds}\right| = \left|\frac{d\vec{t}\,(t)}{ds}\right| = \left|\frac{d\vec{t}\,(t)}{dt}\frac{dt}{ds}\right| = \frac{|\vec{t}\,'(t)|}{|\vec{r}\,'(t)|}$$

證明： $\kappa(t)=\dfrac{|\vec{t}\,'(t)|}{|\vec{r}\,'(t)|}=\dfrac{|\vec{r}\,'\times\vec{r}\,''|}{(\vec{r}\,'\cdot\vec{r}\,')^{\frac{3}{2}}}$ **與** $\tau=\dfrac{[\vec{r}\,'\ \vec{r}\,''\ \vec{r}\,''']}{|\vec{r}\,'\times\vec{r}\,''|^2}$

Proof:

(1) $$\frac{d\vec{r}(t)}{ds}=\frac{d\vec{r}(t)}{dt}\cdot\frac{dt}{ds}=\frac{\vec{r}\,'(t)}{\sqrt{\vec{r}\,'(t)\cdot\vec{r}\,'(t)}}$$

$$\frac{d^2\vec{r}(t)}{ds^2}=\frac{d}{dt}\left(\frac{d\vec{r}(t)}{ds}\right)\cdot\frac{dt}{ds}=\frac{1}{\sqrt{\vec{r}\,'(t)\cdot\vec{r}\,'(t)}}\left[\frac{\vec{r}\,''(t)}{\sqrt{\vec{r}\,'(t)\cdot\vec{r}\,'(t)}}-\frac{\vec{r}\,'(t)\cdot\vec{r}\,''(t)}{(\vec{r}\,'\cdot\vec{r}\,')^{\frac{3}{2}}}\vec{r}\,'(t)\right]$$

$$=\left[\frac{\vec{r}\,''(t)}{\vec{r}\,'(t)\cdot\vec{r}\,'(t)}-\frac{\vec{r}\,'(t)\cdot\vec{r}\,''(t)}{(\vec{r}\,'\cdot\vec{r}\,')^{2}}\vec{r}\,'(t)\right]$$

$$\kappa^2=\frac{d^2\vec{r}(t)}{ds^2}\cdot\frac{d^2\vec{r}(t)}{ds^2}=\left[\frac{\vec{r}\,''(t)}{\vec{r}\,'(t)\cdot\vec{r}\,'(t)}-\frac{\vec{r}\,'(t)\cdot\vec{r}\,''(t)}{(\vec{r}\,'\cdot\vec{r}\,')^{2}}\vec{r}\,'(t)\right]\cdot\left[\frac{\vec{r}\,''(t)}{\vec{r}\,'(t)\cdot\vec{r}\,'(t)}-\frac{\vec{r}\,'(t)\cdot\vec{r}\,''(t)}{(\vec{r}\,'\cdot\vec{r}\,')^{2}}\vec{r}\,'(t)\right]$$

$$=\frac{[\vec{r}\,'(t)\cdot\vec{r}\,'(t)]\,[\vec{r}\,''(t)\cdot\vec{r}\,''(t)]-[\vec{r}\,'(t)\cdot\vec{r}\,''(t)]^2}{[\vec{r}\,'(t)\cdot\vec{r}\,'(t)]^3}$$

由 Lagrange Identity: $|\vec{A}\times\vec{B}|=\sqrt{(\vec{A}\cdot\vec{A})(\vec{B}\cdot\vec{B})-(\vec{A}\cdot\vec{B})^2}$ 可知

$$\kappa^2=\frac{|\vec{r}\,'(t)\times\vec{r}\,''(t)|^2}{[\vec{r}\,'(t)\cdot\vec{r}\,'(t)]^3}\quad\Rightarrow\kappa=\frac{|\vec{r}\,'(t)\times\vec{r}\,''(t)|}{[\vec{r}\,'(t)\cdot\vec{r}\,'(t)]^{\frac{3}{2}}}$$

(2) $$\vec{t}(t)=\frac{1}{\sqrt{\vec{r}\,'(t)\cdot\vec{r}\,'(t)}}\vec{r}\,'(t)$$

$$\vec{n}(t)=\frac{1}{\kappa}\frac{d^2\vec{r}(t)}{ds^2}=\frac{1}{\kappa}\frac{[\vec{r}\,'(t)\cdot\vec{r}\,'(t)]\,\vec{r}\,''(t)-[\vec{r}\,'(t)\cdot\vec{r}\,''(t)]\vec{r}\,'(t)}{[\vec{r}\,'(t)\cdot\vec{r}\,'(t)]^2}$$

又 $\vec{b}(t)=\vec{t}(t)\times\vec{n}(t)=\dfrac{1}{\kappa}\dfrac{\vec{r}\,'(t)\times\vec{r}\,''(t)}{[\vec{r}\,'(t)\cdot\vec{r}\,'(t)]^{\frac{3}{2}}}=\dfrac{\vec{r}\,'(t)\times\vec{r}\,''(t)}{|\vec{r}\,'(t)\times\vec{r}\,''(t)|}$

令 $\phi(t)=|\vec{r}\,'(t)\times\vec{r}\,''(t)|$

$$\frac{d\vec{b}(t)}{ds}=\frac{d\vec{b}(t)}{dt}\cdot\frac{dt}{ds}=\frac{1}{\sqrt{\vec{r}\,'(t)\cdot\vec{r}\,'(t)}}\left[\frac{\vec{r}\,'(t)\times\vec{r}\,'''(t)}{\phi(t)}-\frac{\phi'(t)}{[\phi(t)]^2}(\vec{r}\,'(t)\times\vec{r}\,''(t))\right]$$

由 $\tau=\vec{n}(t)\cdot\dfrac{d\vec{b}(t)}{ds}$ 可得

$$\tau = \frac{1}{\kappa}\frac{[\vec{r}\,'(t)\cdot\vec{r}\,'(t)]\,\vec{r}\,''(t) - [\vec{r}\,'(t)\cdot\vec{r}\,''(t)]\,\vec{r}\,'(t)}{[\vec{r}\,'(t)\cdot\vec{r}\,'(t)]^2}\cdot\frac{1}{\sqrt{\vec{r}\,'(t)\cdot\vec{r}\,'(t)}}\left[\frac{\vec{r}\,'(t)\times\vec{r}\,'''(t)}{\phi(t)} - \frac{\phi'(t)}{[\phi(t)]^2}(\vec{r}\,'(t)\times\vec{r}\,''(t))\right]$$

$$= \frac{1}{\kappa}\frac{-[\vec{r}\,'(t)\cdot\vec{r}\,'(t)]\,\vec{r}\,''(t)}{[\vec{r}\,'(t)\cdot\vec{r}\,'(t)]^2}\cdot\frac{\vec{r}\,'(t)\times\vec{r}\,'''(t)}{\phi(t)\sqrt{\vec{r}\,'(t)\cdot\vec{r}\,'(t)}}$$

$$= \frac{[\vec{r}\,'\ \vec{r}\,''\ \vec{r}\,''']}{\left|\vec{r}\,'\times\vec{r}\,''\right|^2}$$

Frenet 定理:

向量表示式:

$$\frac{d}{ds}\begin{Bmatrix}\vec{T}(s)\\ \vec{N}(s)\\ \vec{B}(s)\end{Bmatrix}=\begin{bmatrix}0 & \kappa(s) & 0\\ -\kappa(s) & 0 & \tau(s)\\ 0 & -\tau(s) & 0\end{bmatrix}\begin{Bmatrix}\vec{T}(s)\\ \vec{N}(s)\\ \vec{B}(s)\end{Bmatrix}$$

分量表示式:

$$\vec{T}(s)=T_1(s)\,\hat{i}+T_2(s)\,\hat{j}+T_3(s)\,\hat{k}$$

$$\vec{N}(s)=N_1(s)\,\hat{i}+N_2(s)\,\hat{j}+N_3(s)\,\hat{k}$$

$$\vec{B}(s)=B_1(s)\,\hat{i}+B_2(s)\,\hat{j}+B_3(s)\,\hat{k}$$

$$\frac{d}{ds}\begin{Bmatrix}T_1(s)\\ T_2(s)\\ T_3(s)\\ N_1(s)\\ N_2(s)\\ N_3(s)\\ B_1(s)\\ B_2(s)\\ B_3(s)\end{Bmatrix}=\left[\begin{array}{ccc:ccc:ccc}0&0&0&\kappa&0&0&0&0&0\\ 0&0&0&0&\kappa&0&0&0&0\\ 0&0&0&0&0&\kappa&0&0&0\\ \hdashline -\kappa&0&0&0&0&0&\tau&0&0\\ 0&-\kappa&0&0&0&0&0&\tau&0\\ 0&0&-\kappa&0&0&0&0&0&\tau\\ \hdashline 0&0&0&-\tau&0&0&0&0&0\\ 0&0&0&0&-\tau&0&0&0&0\\ 0&0&0&0&0&-\tau&0&0&0\end{array}\right]\begin{Bmatrix}T_1(s)\\ T_2(s)\\ T_3(s)\\ N_1(s)\\ N_2(s)\\ N_3(s)\\ B_1(s)\\ B_2(s)\\ B_3(s)\end{Bmatrix}$$

將上式化簡

由 $\frac{d}{ds}\begin{Bmatrix}T_1(s)\\ N_1(s)\\ B_1(s)\end{Bmatrix}=\begin{bmatrix}0&\kappa&0\\ -\kappa&0&\tau\\ 0&-\tau&0\end{bmatrix}\begin{Bmatrix}T_1(s)\\ N_1(s)\\ B_1(s)\end{Bmatrix}$

令 $P(s)=\begin{Bmatrix}T_1(s)\\ N_1(s)\\ B_1(s)\end{Bmatrix}$, $A=\begin{bmatrix}0&\kappa(s)&0\\ -\kappa(s)&0&\tau(s)\\ 0&-\tau(s)&0\end{bmatrix}$ $\Rightarrow P'=AP$

$$\Rightarrow \begin{Bmatrix}T_1(s)\\ N_1(s)\\ B_1(s)\end{Bmatrix}=e^{As}\begin{Bmatrix}T_1(0)\\ N_1(0)\\ B_1(0)\end{Bmatrix}$$

Frenet 定理補充算例

正算問題

已知 $x(t)=a\cos t$, $y(t)=a\sin t$ 與 $z(t)=bt$，求 $\vec{T}$、$\vec{N}$、$\vec{B}$、κ 與 τ。

$$ds=\sqrt{(dx)^2+(dy)^2+(dx)^2}$$
$$=\sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2+(\frac{dz}{dt})^2}\ dt$$
$$\sqrt{(-a\sin t)^2+(a\cos t)^2+(b)^2}\ dt$$
$$=\sqrt{a^2+b^2}\ dt$$

$$\Rightarrow s=\sqrt{a^2+b^2}\,t$$

將時間參數 (t) 轉換成空間參數 (s)

$$\therefore\quad X(s)=a\cos(\frac{s}{\sqrt{a^2+b^2}}),\ Y(s)=a\sin(\frac{s}{\sqrt{a^2+b^2}})\ \text{與}\ Z(s)=\frac{bs}{\sqrt{a^2+b^2}}$$

令 $c=\sqrt{a^2+b^2}$

$$\vec{R}(s)=a\cos(\frac{s}{c})\vec{i}+a\sin(\frac{s}{c})\vec{j}+\frac{bs}{c}\vec{k}$$

$$\vec{T}(s)=\frac{d\vec{R}(s)}{ds}=\vec{R}'(s)=-\frac{a}{c}\sin(\frac{s}{c})\vec{i}+\frac{a}{c}\cos(\frac{s}{c})\vec{j}+\frac{b}{c}\vec{k}$$

$$\vec{N}(s)=\frac{\vec{T}'(s)}{\left|\vec{T}'(s)\right|}=\frac{-\frac{a}{c^2}\cos(\frac{s}{c})\vec{i}-\frac{a}{c^2}\sin(\frac{s}{c})\vec{j}+0\vec{k}}{\frac{a}{c^2}}=-\cos(\frac{s}{c})\vec{i}-\sin(\frac{s}{c})\vec{j}$$

$$\vec{B}(s)=\vec{T}(s)\times\vec{N}(s)=\frac{b}{c}\sin(\frac{s}{c})\vec{i}-\frac{b}{c}\cos(\frac{s}{c})\vec{j}+\frac{a}{c}\vec{k}$$

$$\vec{B}'(s)=\frac{d\vec{B}(s)}{ds}=\frac{b}{c^2}\cos(\frac{s}{c})\vec{i}+\frac{b}{c^2}\sin(\frac{s}{c})\vec{j}+0\vec{k}$$

$$\kappa(s)=\left|\vec{T}'(s)\right|=\frac{a}{c^2}=\frac{a}{a^2+b^2}\qquad\Rightarrow\rho(s)=\frac{1}{\kappa(s)}=\frac{a^2+b^2}{a}$$

$$\tau(s)=\left|\vec{B}'(s)\right|=\frac{b}{c^2}=\frac{b}{a^2+b^2}\qquad\Rightarrow\sigma(s)=\frac{1}{\tau(s)}=\frac{a^2+b^2}{b}$$

當 $s=0$

$$\vec{T}(0)=(0,\frac{a}{c},\frac{b}{c}),\ \ \vec{N}(0)=(-1,0,0),\ \ \vec{B}(s)=(0,-\frac{b}{c},\frac{a}{c})$$

$$X(0)=1,\ Y(0)=0 \ \text{與}\ Z(0)=0$$

■ 已知 $x(t)=\cos t$, $y(t)=\sin t$ 與 $z(t)=t$，求 $\vec{T}$、$\vec{N}$、$\vec{B}$、κ 與 τ。

$$ds=\sqrt{2}\,dt \quad \Rightarrow s=\sqrt{2}\,t$$

將時間參數 (t) 轉換成空間參數 (s)

$$\therefore X(s)=\cos(\frac{s}{\sqrt{2}}),\ Y(s)=\sin(\frac{s}{\sqrt{2}}) \ \text{與}\ Z(s)=\frac{s}{\sqrt{2}}$$

$$\vec{R}(s)=\cos(\frac{s}{\sqrt{2}})\vec{i}+\sin(\frac{s}{\sqrt{2}})\vec{j}+\frac{s}{\sqrt{2}}\vec{k}$$

$$\vec{T}(s)=\frac{d\vec{R}(s)}{ds}=\vec{R}'(s)=-\frac{1}{\sqrt{2}}\sin(\frac{s}{\sqrt{2}})\vec{i}+\frac{1}{\sqrt{2}}\cos(\frac{s}{\sqrt{2}})\vec{j}+\frac{1}{\sqrt{2}}\vec{k}$$

$$\vec{N}(s)=\frac{\vec{T}'(s)}{\left|\vec{T}'(s)\right|}=-\cos(\frac{s}{\sqrt{2}})\vec{i}-\sin(\frac{s}{\sqrt{2}})\vec{j}$$

$$\vec{B}(s)=\vec{T}(s)\times\vec{N}(s)=\frac{1}{\sqrt{2}}\sin(\frac{s}{\sqrt{2}})\vec{i}-\frac{1}{\sqrt{2}}\cos(\frac{s}{\sqrt{2}})\vec{j}+\frac{1}{\sqrt{2}}\vec{k}$$

$$\kappa(s)=\left|\vec{T}'(s)\right|=\frac{1}{2} \qquad \Rightarrow \rho(s)=\frac{1}{\kappa(s)}=2$$

$$\tau(s)=\left|\vec{B}'(s)\right|=\frac{1}{2} \qquad \Rightarrow \sigma(s)=\frac{1}{\tau(s)}=2$$

當 $s=0$

$$\vec{T}(0)=(0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}),\ \ \vec{N}(0)=(-1,0,0),\ \ \vec{B}(s)=(0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$$

$$X(0)=1,\ Y(0)=0 \ \text{與}\ Z(0)=0$$

反算問題

已知 $\kappa(s)=\frac{2}{5}$ 與 $\tau(s)=\frac{1}{5}$ 並且知道在點 $(2,0,0)$ 其 $\vec{T}(0)=(0,\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}})$、$\vec{N}(0)=(-1,0,0)$ 與 $\vec{B}(s)=(0,-\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}})$，試問該曲線表示式為何？

由 Frenet 定理可知:

$$\frac{d}{ds}\begin{Bmatrix}\vec{T}(s)\\ \vec{N}(s)\\ \vec{B}(s)\end{Bmatrix}=\begin{bmatrix}0 & \kappa(s) & 0\\ -\kappa(s) & 0 & \tau(s)\\ 0 & -\tau(s) & 0\end{bmatrix}\begin{Bmatrix}\vec{T}(s)\\ \vec{N}(s)\\ \vec{B}(s)\end{Bmatrix}$$

令 $P(s)=\begin{Bmatrix}\vec{T}(s)\\ \vec{N}(s)\\ \vec{B}(s)\end{Bmatrix} \qquad \Rightarrow \vec{P}'(s)=A(s)\vec{P}(s)$

$$A=\begin{bmatrix}0 & \frac{2}{\sqrt{5}} & 0\\ -\frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}}\\ 0 & -\frac{1}{\sqrt{5}} & 0\end{bmatrix}$$

$$\det(A-\lambda I)=\begin{bmatrix}-\lambda & \frac{2}{\sqrt{5}} & 0\\ -\frac{2}{\sqrt{5}} & -\lambda & \frac{1}{\sqrt{5}}\\ 0 & -\frac{1}{\sqrt{5}} & -\lambda\end{bmatrix}=0 \quad \Rightarrow \lambda^3+\lambda=0 \quad \Rightarrow \lambda=0,\ \lambda=\pm i$$

$$\lambda_1=0 \quad \Rightarrow v_1=\begin{Bmatrix}1\\ 0\\ 2\end{Bmatrix}$$

$$\lambda_2=i \quad \Rightarrow v_2=\begin{Bmatrix}\frac{2}{\sqrt{5}}\\ i\\ -\frac{1}{\sqrt{5}}\end{Bmatrix}$$

$$\lambda_3=-i \quad \Rightarrow v_3=\begin{Bmatrix}-\frac{2}{\sqrt{5}}\\ i\\ \frac{1}{\sqrt{5}}\end{Bmatrix}$$

$$Q=\begin{bmatrix}1 & \frac{2}{\sqrt{5}} & -\frac{2}{\sqrt{5}}\\ 0 & i & i\\ 2 & -\frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{bmatrix},\quad D=\begin{bmatrix}0 & 0 & 0\\ 0 & i & 0\\ 0 & 0 & -i\end{bmatrix},\quad Q^{-1}=\begin{bmatrix}\frac{1}{5} & 0 & \frac{2}{5}\\ \frac{1}{\sqrt{5}} & -\frac{i}{2} & -\frac{1}{2\sqrt{5}}\\ -\frac{1}{\sqrt{5}} & -\frac{i}{2} & \frac{1}{2\sqrt{5}}\end{bmatrix}$$

$$\vec{P}(s)=e^{As}\cdot\vec{P}(0)$$

$$=Qe^{Ds}Q^{-1}\begin{Bmatrix}0\vec{i}+\frac{2}{\sqrt{5}}\vec{j}+\frac{1}{\sqrt{5}}\vec{k}\\ -\vec{i}+0\vec{j}+0\vec{k}\\ 0\vec{i}-\frac{1}{\sqrt{5}}\vec{j}+\frac{2}{\sqrt{5}}\vec{k}\end{Bmatrix}$$

$$=\begin{bmatrix}\frac{2}{5}(e^{-is}+e^{is}) & \frac{i}{\sqrt{5}}(e^{-is}-e^{is}) & -\frac{1}{5}(e^{-is}+e^{is})\\ -\frac{i}{\sqrt{5}}(e^{-is}-e^{is}) & \frac{1}{2}(e^{-is}+e^{is}) & \frac{i}{2\sqrt{5}}(e^{-is}-e^{is})\\ -\frac{1}{5}(e^{-is}+e^{is}) & -\frac{i}{2\sqrt{5}}(e^{-is}-e^{is}) & \frac{1}{10}(e^{-is}+e^{is})\end{bmatrix}\begin{Bmatrix}\frac{1}{\sqrt{5}}(2\vec{j}+\vec{k})\\ -\vec{i}\\ -\frac{1}{\sqrt{5}}(\vec{j}-2\vec{k})\end{Bmatrix}$$

$$=\begin{bmatrix}\frac{4}{5}\cos(s) & \frac{2}{\sqrt{5}}\sin(s) & -\frac{2}{5}\cos(s)\\ -\frac{2}{\sqrt{5}}\sin(s) & \cos(s) & \frac{1}{\sqrt{5}}\sin(s)\\ -\frac{2}{5}\cos(s) & -\frac{1}{\sqrt{5}}\sin(s) & \frac{1}{5}\cos(s)\end{bmatrix}\begin{Bmatrix}\frac{1}{\sqrt{5}}(2\vec{j}+\vec{k})\\ -\vec{i}\\ -\frac{1}{\sqrt{5}}(\vec{j}-2\vec{k})\end{Bmatrix}$$

$$=\begin{Bmatrix}\frac{4}{5\sqrt{5}}\cos(s)\cdot(2\vec{j}+\vec{k})-\frac{2}{\sqrt{5}}\sin(s)\cdot(\vec{i})+\frac{2}{5\sqrt{5}}\cos(s)\cdot(\vec{j}-2\vec{k})\\ -\frac{1}{5}\sin(s)\cdot(2\vec{j}+\vec{k})-\cos(s)\cdot(\vec{i})-\frac{1}{5}\sin(s)\cdot(\vec{j}-2\vec{k})\\ -\frac{2}{5\sqrt{5}}\cos(s)\cdot(2\vec{j}+\vec{k})+\frac{1}{\sqrt{5}}\sin(s)\cdot(\vec{i})-\frac{1}{5\sqrt{5}}\cos(s)\cdot(\vec{j}-2\vec{k})\end{Bmatrix}$$

$$=\begin{Bmatrix}-\frac{2}{\sqrt{5}}\sin(s)\;\vec{i}+\frac{2}{\sqrt{5}}\cos(s)\;\vec{j}+0\vec{k}\\ -\cos(s)\;\vec{i}-\frac{3}{5}\sin(s)\;\vec{j}+\frac{1}{5}\sin(s)\vec{k}\\ \frac{1}{\sqrt{5}}\sin(s)\;\vec{i}-\frac{1}{\sqrt{5}}\cos(s)\;\vec{j}+0\vec{k}\end{Bmatrix}$$

[illegible]

所以可得

$$\vec{T}(s) = -\frac{2}{\sqrt{5}}\sin(s)\ \vec{i} + \frac{2}{\sqrt{5}}\cos(s)\ \vec{j} + 0\vec{k}$$

$$\vec{N}(s) = -\cos(s)\ \vec{i} - \frac{3}{5}\sin(s)\ \vec{j} + \frac{1}{5}\sin(s)\vec{k}$$

$$\vec{B}(s) = \frac{1}{\sqrt{5}}\sin(s)\ \vec{i} - \frac{1}{\sqrt{5}}\cos(s)\ \vec{j} + 0\vec{k}$$

$$\vec{R}(s) = a\cos(\frac{s}{c})\vec{i} + a\sin(\frac{s}{c})\ \vec{j} + \frac{bs}{c}\vec{k}$$

$$\vec{T}(s) = \frac{d\vec{R}(s)}{ds} = \vec{R}'(s) = -\frac{a}{c}\sin(\frac{s}{c})\vec{i} + \frac{a}{c}\cos(\frac{s}{c})\ \vec{j} + \frac{b}{c}\vec{k}$$

$$\vec{N}(s) = \frac{\vec{T}'(s)}{\left|\vec{T}'(s)\right|} = \frac{-\frac{a}{c^2}\cos(\frac{s}{c})\vec{i} - \frac{a}{c^2}\sin(\frac{s}{c})\ \vec{j} + 0\vec{k}}{\frac{a}{c^2}} = -\cos(\frac{s}{c})\vec{i} - \sin(\frac{s}{c})\ \vec{j}$$

$$\vec{B}(s) = \vec{T}(s) \times \vec{N}(s) = \frac{b}{c}\sin(\frac{s}{c})\vec{i} - \frac{b}{c}\cos(\frac{s}{c})\ \vec{j} + \frac{a}{c}\vec{k}$$

$$\vec{B}'(s) = \frac{d\vec{B}(s)}{ds} = \frac{b}{c^2}\cos(\frac{s}{c})\vec{i} + \frac{b}{c^2}\sin(\frac{s}{c})\ \vec{j} + 0\vec{k}$$

$$\kappa(s) = \left|\vec{T}'(s)\right| = \frac{a}{c^2} = \frac{a}{a^2+b^2} \qquad \Rightarrow \rho(s) = \frac{1}{\kappa(s)} = \frac{a^2+b^2}{a}$$

$$\tau(s) = \left|\vec{B}'(s)\right| = \frac{b}{c^2} = \frac{b}{a^2+b^2} \qquad \Rightarrow \sigma(s) = \frac{1}{\tau(s)} = \frac{a^2+b^2}{b}$$

與正算結果比較後可知所得與正算結果相同

由於 $\dfrac{d\vec{R}(s)}{ds}=\vec{T}(s)=-\dfrac{1}{\sqrt{2}}\sin(\dfrac{s}{\sqrt{2}})\vec{i}+\dfrac{1}{\sqrt{2}}\cos(\dfrac{s}{\sqrt{2}})\vec{j}+\dfrac{1}{\sqrt{2}}\vec{k}$

$$\Rightarrow \vec{R}(s)=[\cos(\frac{s}{\sqrt{2}})+c_1]\vec{i}+[\sin(\frac{s}{\sqrt{2}})+c_2]\vec{j}+[\frac{s}{\sqrt{2}}+c_3]\vec{k}$$

又 $X(0)=1$，$Y(0)=0$ 與 $Z(0)=0$

$X(s)=\cos(\dfrac{s}{\sqrt{2}})+c_1$， $X(0)=1 \quad \Rightarrow c_1=0$

$Y(s)=\sin(\dfrac{s}{\sqrt{2}})+c_2$， $Y(0)=0 \quad \Rightarrow c_2=0$

$Z(s)=\dfrac{s}{\sqrt{2}}+c_3$， $Z(0)=0 \quad \Rightarrow c_3=0$

$\therefore \quad \vec{R}(s)=\cos(\dfrac{s}{\sqrt{2}})\vec{i}+\sin(\dfrac{s}{\sqrt{2}})\vec{j}+\dfrac{s}{\sqrt{2}}\vec{k}$