# 國立臺灣海洋大學河海工程學系 2025 工程數學(二) 期末考參考解答

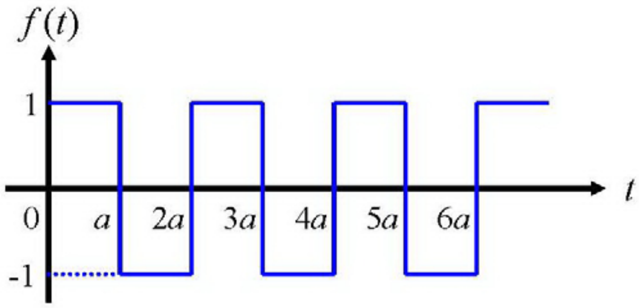
系級：________　　學號：________　　姓名：________

1. 試求下列 $f(t)$ 的拉普拉斯轉換為何? (24%)

   (1) $f(t)=\sin^2 t$　(2) $f(t)=\sin^3 t$　(3) $f(t)=\sin 2t\cdot\cos t$

   (4) $f(t)=\sin 2t\cdot\cosh 2t$　(5) $f(t)=\sin(t+\frac{\pi}{3})$　(6) $f(t)=\frac{1}{t}\sin t$

2. 試求下圖函數的拉普拉斯轉換? (10%)




3. 試求下列 $F(s)$ 的拉普拉斯逆轉換為何? (24%)

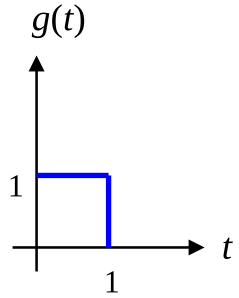
   (1) $F(s)=e^{-3s}$　(2) $F(s)=\dfrac{1}{s^3(s^2+1)}$　(3) $F(s)=\dfrac{2s}{(s^2+9)^2}$

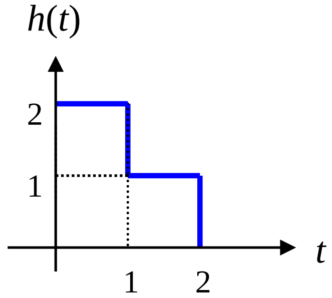
   (4) $F(s)=\dfrac{1}{s^2+4s+13}$　(5) $F(s)=\dfrac{e^{-\pi s}}{s^2+1}$　(6) $F(s)=\ln\dfrac{s^2+4}{s^2}$

4. 單位步階函數(unit step function)定義為

$$u(t-a)=\begin{cases}1, & t>a\\ 0, & t<a\end{cases}\qquad \text{其中 } a \text{ 為常數}$$

   (1) 試將下圖函數 $g(t)$ 與 $h(t)$ 以單位步階函數表示。 (6%)
   (2) 試求 $g(t)$ 與 $h(t)$ 之拉普拉斯轉換 $G(s)$ 與 $H(s)$。 (6%)
   (3) 已知 $F(s)=G(s)\cdot H(s)$ 又 $f(t)=\mathcal{L}^{-1}[F(s)]$，試求 $f(t)$ 並繪其圖形。 (10%)




5. 試以 Laplace 轉換法求下述方程式

   (1) $t-2y(t)=\int_0^t (e^{\tau}-e^{-\tau})y(t-\tau)\,d\tau$ (10%)

   (2) $y'(t)+2e^t\int_0^t y(\tau)e^{-\tau}\,d\tau+y(t)=t^2e^t+2te^t$，$y(0)=-1$ (10%)

   (3) $y''(t)-4y'(t)+13y(t)=5\delta(t-4)$，$y(0)=0$，$y'(0)=0$ (10%)

6. 試使用拉普拉斯轉換求解下述聯立微分方程組 (10%)

$$\begin{cases} x''(t)+2x'(t)+\int_0^t y(\tau)\,d\tau=0 \\ 4x''(t)-x'(t)+y(t)=e^{-t} \end{cases} \quad 且 \quad x(0)=0, \quad x'(0)=1$$

7. 針對工數這門課的教學方式與如何學好這門課有何感想與建議? (10%)

**[參考公式]**

**拉普拉斯轉換：** $F(s)=\mathcal{L}[f(t)]=\int_0^{\infty} f(t)\,e^{-st}\,dt$

**第一平移定理：** $\mathcal{L}[e^{at}f(t)]=F(s-a)$

**第二平移定理：** $\mathcal{L}[f(t-a)u(t-a)]=e^{-as}F(s)$

**尺度變換：** $\mathcal{L}[f(at)]=\frac{1}{a}F(\frac{s}{a})$

**微分函數的拉普拉斯轉換：** $\mathcal{L}[f'(t)]=sF(s)-f(0)$

$\mathcal{L}[f''(t)]=s^2F(s)-sf(0)-f'(0)$

**積分函數的拉普拉斯轉換：** $\mathcal{L}[\int_0^t f(x)\,dx]=\frac{F(s)}{s}$， $\mathcal{L}[\int_0^t\int_0^{\tau} f(x)\,dx\,d\tau]=\frac{F(s)}{s^2}$

**拉普拉斯轉換的微分：** $\mathcal{L}[tf(t)]=(-1)\frac{d}{ds}F(s)$， $\mathcal{L}[t^2f(t)]=(-1)^2\frac{d^2}{ds^2}F(s)$

**拉普拉斯轉換的積分：** $\mathcal{L}[\frac{f(t)}{t}]=\int_s^{\infty} F(\tau)\,d\tau$， $\mathcal{L}[\frac{f(t)}{t^2}]=\int_s^{\infty}\int_{\gamma}^{\infty} F(\tau)\,d\tau\,d\gamma$

**摺積：** $f(t)*g(t)=\int_0^t f(\tau)\,g(t-\tau)\,d\tau=\int_0^t f(t-\tau)\,g(\tau)\,d\tau$

$\mathcal{L}[f(t)*g(t)]=F(s)\cdot G(s)$

**雙曲函數：** $\cosh at=\frac{e^{at}+e^{-at}}{2}$， $\sinh at=\frac{e^{at}-e^{-at}}{2}$

**初值定理：** $\lim_{t\to 0} f(t)=\lim_{s\to\infty} sF(s)$

**終值定理：** $\lim_{t\to\infty} f(t)=\lim_{s\to 0} sF(s)$

**一階線性 ODE:** $y'(x)+p(x)y(x)=q(x)$ **其積分因子為** $\mu=e^{\int p(x)dx}$

**[參考解答]**

**1. 試求下列 $f(t)$ 的拉普拉斯轉換為何? (24%)**

**(1)** $f(t)=\sin^2 t$ **(2)** $f(t)=\sin^3 t$ **(3)** $f(t)=\sin 2t\cdot\cos t$

**(4)** $f(t)=\sin 2t\cdot\cosh 2t$ **(5)** $f(t)=\sin(t+\frac{\pi}{3})$ **(6)** $f(t)=\frac{1}{t}\sin t$

(1) $\mathcal{L}[\sin^2 t]=\mathcal{L}[\frac{1-\cos 2t}{2}]=\frac{1}{2}(\frac{1}{s}-\frac{s}{s^2+4})=\frac{2}{s(s^2+4)}$

(2) $\sin 3t=3\sin t-4\sin^3 t \quad \Rightarrow \sin^3 t=\frac{1}{4}(3\sin t-\sin 3t)$

$$\mathcal{L}[\sin^3 t]=\mathcal{L}[\frac{1}{4}(3\sin t-\sin 3t)]=\frac{1}{4}(\frac{3}{s^2+1}-\frac{3}{s^2+9})=\frac{6}{(s^2+1)(s^2+9)}$$

(3) $\sin 2t\cdot\cos t=\frac{1}{2}[\sin(2t+t)+\sin(2t-t)]=\frac{1}{2}(\sin 3t+\sin t)$

$$\mathcal{L}[\sin 2t\cdot\cos t]=\mathcal{L}[\frac{1}{2}(\sin 3t+\sin t)]=\frac{1}{2}(\frac{3}{s^2+9}+\frac{1}{s^2+1})=\frac{2s^2+6}{(s^2+1)(s^2+9)}$$

(4) $\mathcal{L}[\sin 2t\cdot\cosh 2t]=\frac{1}{2}\mathcal{L}[(e^{2t}\sin 2t+e^{-2t}\sin 2t)]=\frac{1}{2}[\frac{2}{(s-2)^2+4}+\frac{2}{(s+2)^2+4}]$

$$=\frac{1}{(s-2)^2+4}+\frac{1}{(s+2)^2+4}$$

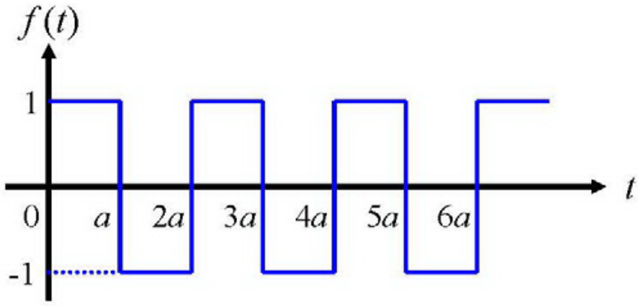
(5) $f(t)=\sin(t+\frac{\pi}{3})=\sin t\cdot\cos\frac{\pi}{3}+\cos t\cdot\sin\frac{\pi}{3}=\frac{1}{2}\sin t+\frac{\sqrt{3}}{2}\cos t)$

$$\therefore\ \mathcal{L}[\sin(t+\frac{\pi}{3})]=\frac{1}{2}\frac{1}{s^2+1}+\frac{\sqrt{3}}{2}\frac{s}{s^2+1^2})=\frac{\sqrt{3}s+1}{2(s^2+1)}$$

(6) $\mathcal{L}[\sin t]=\frac{1}{s^2+1}$

$$\therefore\ \mathcal{L}[\frac{1}{t}\sin t]=\int_s^\infty\frac{1}{\tau^2+1}d\tau=\tan^{-1}\tau\Big|_s^\infty=\frac{\pi}{2}-\tan^{-1}s$$

**2. 試求下圖函數的拉普拉斯轉換? (10%)**




$$T = 2a$$

$$\begin{aligned}
\mathcal{L}[f(t)] &= \frac{1}{1-e^{-2as}}\int_0^{2a} f(t)e^{-st}dt \\
&= \frac{1}{1-e^{-2as}}\left[\int_0^{a} 1\cdot e^{-st}dt + \int_a^{2a} (-1)\cdot e^{-st}dt\right] \\
&= \frac{1}{1-e^{-2as}}\left[-\frac{1}{s}e^{-st}\Big|_0^a + \frac{1}{s}e^{-st}\Big|_a^{2a}\right] \\
&= \frac{1}{1-e^{-2as}}\left[-\frac{1}{s}e^{-sa} + \frac{1}{s} + \frac{1}{s}e^{-2as} - \frac{1}{s}e^{-sa}\right] \\
&= \frac{(1-e^{-as})^2}{s(1-e^{-2as})} \\
&= \frac{1-e^{-as}}{s(1+e^{-as})}
\end{aligned}$$

**3. 試求下列 $F(s)$ 的拉普拉斯逆轉換為何? (24%)**

**(1)** $F(s)=e^{-3s}$ **(2)** $F(s)=\dfrac{1}{s^3(s^2+1)}$ **(3)** $F(s)=\dfrac{2s}{(s^2+9)^2}$

**(4)** $F(s)=\dfrac{1}{s^2+4s+13}$ **(5)** $F(s)=\dfrac{e^{-\pi s}}{s^2+1}$ **(6)** $F(s)=\ln\dfrac{s^2+4}{s^2}$

(1) $F(s)=e^{-3s} \quad \Rightarrow \mathcal{L}^{-1}[F(s)]=\delta(t-3)$

(2) $F(s)=\dfrac{1}{s^3(s^2+1)}=-\dfrac{1}{s}+\dfrac{1}{s^3}+\dfrac{s}{s^2+1} \quad \Rightarrow \mathcal{L}^{-1}[F(s)]=-1+\dfrac{1}{2}t^2+\cos t$

(3) $F(s)=\dfrac{2s}{(s^2+9)^2}=-\dfrac{d}{ds}(\dfrac{1}{s^2+9})=-\dfrac{1}{3}\dfrac{d}{ds}(\dfrac{3}{s^2+3^2})$

$\Rightarrow \mathcal{L}^{-1}[F(s)]=\dfrac{1}{3}\mathcal{L}^{-1}[-\dfrac{d}{ds}(\dfrac{3}{s^2+3^2})]=\dfrac{1}{3}t\cdot\mathcal{L}^{-1}[\dfrac{3}{s^2+3^2}]=\dfrac{1}{3}t\sin 3t$

(4) $F(s)=\dfrac{1}{s^2+4s+13}=\dfrac{1}{(s+2)^2+3^2}=\dfrac{1}{3}\dfrac{3}{(s+2)^2+3^2}$

$\Rightarrow \mathcal{L}^{-1}[F(s)]=\dfrac{1}{3}\mathcal{L}^{-1}[\dfrac{3}{(s+2)^2+3^2}]=\dfrac{1}{3}e^{-2t}\cdot\mathcal{L}^{-1}[\dfrac{3}{s^2+3^2}]=\dfrac{1}{3}e^{-2t}\sin 3t$

(5) $F(s)=\dfrac{e^{-\pi s}}{s^2+1}$

$\because\ f(t)=\mathcal{L}^{-1}[\dfrac{1}{s^2+1}]=\sin t$

$\therefore\ \mathcal{L}^{-1}[F(s)]=\mathcal{L}^{-1}[\dfrac{e^{-\pi s}}{s^2+1}]=u(t-\pi)f(t-\pi)=u(t-\pi)\sin(t-\pi)$

$=-u(t-\pi)\sin t$

(6) $F(s)=\ln\dfrac{s^2+4}{s^2} \quad \Rightarrow \mathcal{L}[f(t)]=\ln\dfrac{s^2+4}{s^2}$

$\Rightarrow \mathcal{L}[t\,f(t)]=-\dfrac{d}{ds}\left[\ln\dfrac{s^2+4}{s^2}\right]=-\dfrac{d}{ds}\left[\ln(s^2+4)-\ln(s^2)\right]$

$\Rightarrow \mathcal{L}[t\,f(t)]=-(\dfrac{2s}{s^2+4}-\dfrac{2}{s})$

$\Rightarrow t\,f(t)=\mathcal{L}^{-1}[\dfrac{2}{s}-\dfrac{2s}{s^2+4}]=2-2\cos 2t$

$\Rightarrow f(t)=\dfrac{2}{t}(1-\cos 2t)$

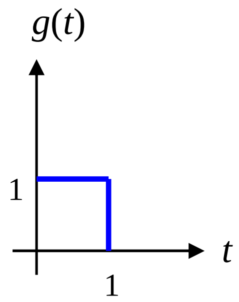
**4. 單位步階函數(unit step function)定義為**

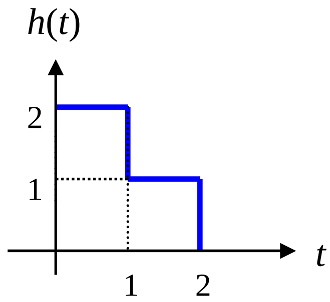
$$u(t-a)=\begin{cases}1, & t>a\\ 0, & t<a\end{cases}\qquad \text{其中 } \boldsymbol{a} \text{ 為常數}$$

**(1)** 試將下圖函數 $g(t)$ 與 $h(t)$ 以單位步階函數表示。 **(6%)**

**(2)** 試求 $g(t)$ 與 $h(t)$ 之拉普拉斯轉換 $G(s)$ 與 $H(s)$ 。 **(6%)**

**(3)** 已知 $F(s)=G(s)\cdot H(s)$ 又 $f(t)=\mathcal{L}^{-1}[F(s)]$，試求 $f(t)$ 並繪其圖形。 **(10%)**




(1) $g(t)=u(t)-u(t-1)$

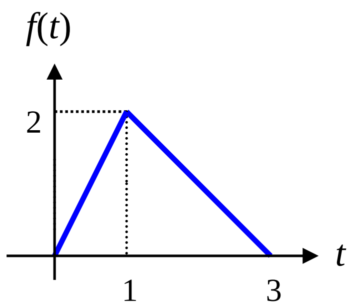
$h(t)=2[u(t)-u(t-1)]+[u(t-1)-u(t-2)]=2u(t)-u(t-1)-u(t-2)$

(2) $\mathcal{L}[g(t)]=G(s)=\dfrac{1}{s}-\dfrac{1}{s}e^{-s}$

$\mathcal{L}[h(t)]=H(s)=\dfrac{2}{s}-\dfrac{1}{s}e^{-s}-\dfrac{1}{s}e^{-2s}$

(3) $F(s)=G(s)\cdot H(s)=\dfrac{1}{s^2}(2-3e^{-s}+e^{-3s})$

$\Rightarrow f(t)=\mathcal{L}^{-1}[F(s)]=g(t)*h(t)=2tu(t)-3(t-1)u(t-1)+(t-3)u(t-3)$



**5. 試以 Laplace 轉換法求下述方程式 (30%)**

(1) $t-2y(t)=\int_0^t (e^{\tau}-e^{-\tau})y(t-\tau)\,d\tau$ (112 台科大機械)

(2) $y'(t)+2e^t\int_0^t y(\tau)e^{-\tau}\,d\tau+y(t)=t^2e^t+2te^t$, $y(0)=-1$ (108 交大機械丙組)

(3) $y''(t)-4y'(t)+13y(t)=5\delta(t-4)$, $y(0)=0$, $y'(0)=0$ (109 台科大營建)

(1) $t-2y(t)=\int_0^t (e^{\tau}-e^{-\tau})y(t-\tau)\,d\tau$

$$\Rightarrow \mathcal{L}[t-2y(t)]=\mathcal{L}[\int_0^t (e^{\tau}-e^{-\tau})y(t-\tau)\,d\tau]$$

$$\Rightarrow \mathcal{L}[t-2y(t)]=\mathcal{L}[(e^{t}-e^{-t})*y(t)]$$

$$\Rightarrow \frac{1}{s^2}-2Y(s)=(\frac{1}{s-1}-\frac{1}{s+1})Y(s)$$

$$\Rightarrow \frac{2s^2}{s^2-1}Y(s)=\frac{1}{s^2}$$

$$\Rightarrow Y(s)=\frac{s^2-1}{2s^4}=\frac{1}{2}(\frac{1}{s^2}-\frac{1}{s^4})$$

$$\mathcal{L}^{-1}[Y(s)]=\mathcal{L}^{-1}[\frac{1}{2}(\frac{1}{s^2}-\frac{1}{s^4})]=\frac{1}{2}(t-\frac{1}{6}t^3)=\frac{1}{2}t-\frac{1}{12}t^3$$

(2) $y'(t)+2e^t\int_0^t y(\tau)e^{-\tau}\,d\tau+y(t)=t^2e^t+2te^t$

$$\Rightarrow \mathcal{L}[y'(t)+2\int_0^t y(\tau)e^{t-\tau}\,d\tau+y(t)]=\mathcal{L}[t^2e^t+2te^t]$$

$$\Rightarrow \mathcal{L}[y'(t)+2y(t)*e^t+y(t)]=\mathcal{L}[t^2e^t+2te^t]$$

$$\Rightarrow sY(s)-y(0)+2Y(s)\cdot\frac{1}{s-1}+Y(s)=\frac{2}{(s-1)^3}+\frac{2}{(s-1)^2}$$

$$\Rightarrow (s+\frac{2}{s-1}+1)Y(s)=\frac{2}{(s-1)^3}+\frac{2}{(s-1)^2}-1$$

$$\Rightarrow (\frac{s^2+1}{s-1})Y(s)=\frac{2s}{(s-1)^3}-1$$

$$\Rightarrow Y(s)=\frac{2s}{(s-1)^2(s^2+1)}-\frac{s-1}{s^2+1}$$

$$\Rightarrow Y(s)=\frac{2s}{(s-1)^2(s^2+1)}-\frac{s-1}{s^2+1}=\frac{-1}{s^2+1}+\frac{1}{(s-1)^2}-\frac{s}{s^2+1}+\frac{1}{s^2+1}$$

$$=\frac{1}{(s-1)^2}-\frac{s}{s^2+1}$$

$$\mathcal{L}^{-1}[Y(s)]=\mathcal{L}^{-1}[\frac{1}{(s-1)^2}-\frac{s}{s^2+1}]=te^t-\cos t$$

(3) $y''(t)-4y'(t)+13y(t)=5\delta(t-4)$ , $y(0)=0$ , $y'(0)=0$ (10%)

$\Rightarrow \mathcal{L}[y''(t)-4y'(t)+13y(t)]=\mathcal{L}[5\delta(t-4)]$

$\Rightarrow s^2Y(s)-sy(0)-y'(0)-4sY(s)+sy(0)+13Y(s)=5e^{-4s}$

$\Rightarrow (s^2-4s+13)Y(s)=5e^{-4s}$

$\Rightarrow Y(s)=\dfrac{5e^{-4s}}{s^2-4s+13}=\dfrac{5e^{-4s}}{(s-2)^2+3^2}$

又 $f(t)=\mathcal{L}^{-1}[\dfrac{1}{(s-2)^2+3^2}]=e^{2t}\cdot\mathcal{L}^{-1}[\dfrac{1}{s^2+3^2}]=\dfrac{1}{3}e^{2t}\sin 3t$

$\mathcal{L}^{-1}[Y(s)]=\mathcal{L}^{-1}[\dfrac{5e^{-4s}}{(s-2)^2+3^2}]=5u(t-4)\cdot f(t-4)=\dfrac{5}{3}u(t-4)\cdot e^{2(t-4)}\sin 3(t-4)$

**6. 試使用拉普拉斯轉換求解下述聯立微分方程組 (10%)　　　(108 交大土木)**

$$\begin{cases} x''(t)+2x'(t)+\int_0^t y(\tau)\,d\tau=0 \\ 4x''(t)-x'(t)+y(t)=e^{-t} \end{cases}$$ 且 $x(0)=0$， $x'(0)=1$

取拉普拉斯轉換後可得

$$\begin{cases} \mathcal{L}^{1}[x''(t)+2x'(t)+\int_0^t y(\tau)\,d\tau]=0 \\ \mathcal{L}^{1}[4x''(t)-x'(t)+y(t)]=\mathcal{L}^{1}[e^{-t}] \end{cases}$$

$$\Rightarrow \begin{cases} [s^2X(s)-sx(0)-x'(0)]+2[sX(s)-x(0)]+\dfrac{1}{s}Y(s)=0 \\ 4[s^2X(s)-sx(0)-x'(0)]-[sX(s)-x(0)]+Y(s)=\dfrac{1}{s+1} \end{cases}$$

$$\Rightarrow \begin{cases} (s^2+2s)X(s)+\dfrac{1}{s}Y(s)=1 & \text{......(1)} \\ (4s^2-s)X(s)+Y(s)=\dfrac{1}{s+1}+4 & \text{......(2)} \end{cases}$$

由(2)可得 $Y(s)=\dfrac{1}{s+1}+4-(4s^2-s)X(s)$ 　代回(1)可得

$$(s^2+2s)X(s)+\frac{1}{s}[\frac{1}{s+1}+4-(4s^2-s)X(s)]=1$$

$$\Rightarrow (s^2-2s+1)X(s)=1-\frac{4s+5}{s(s+1)}$$

$$\Rightarrow X(s)=\frac{s^2-3s-5}{s(s+1)(s-1)^2}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s-1}+\frac{D}{(s-1)^2}$$

通分後可得 $s^2-3s-5=A(s+1)(s-1)^2+Bs(s-1)^2+Cs(s+1)(s-1)+Ds(s+1)$

令 $s=0 \ \Rightarrow A=-5$

令 $s=-1 \ \Rightarrow -1=B\cdot(-1)\cdot(-2)^2 \ \Rightarrow B=\dfrac{1}{4}$

令 $s=1 \ \Rightarrow -7=D\cdot 1\cdot 2 \ \Rightarrow D=-\dfrac{7}{2}$

比較 $s^3$ 項: $A+B+C=0 \ \Rightarrow C=\dfrac{19}{4}$

$$\therefore \quad X(s)=\frac{s^2-3s-5}{s(s+1)(s-1)^2}=\frac{-5}{s}+\frac{\frac{1}{4}}{s+1}+\frac{\frac{19}{4}}{s-1}+\frac{-\frac{7}{2}}{(s-1)^2}$$

$$\Rightarrow Y(s)=\frac{1}{s+1}+4-(4s^2-s)\cdot\frac{s^2-3s-5}{s(s+1)(s-1)^2}=\frac{10s^2+11s}{(s+1)(s-1)^2}$$

$$=\frac{E}{(s+1)}+\frac{F}{(s-1)}+\frac{G}{(s-1)^2}$$

通分後可得 $10s^2+11s=E(s-1)^2+F(s+1)(s-1)+G(s+1)$

令 $s=-1 \quad \Rightarrow E=-\frac{1}{4}$

令 $s=1 \quad \Rightarrow G=\frac{21}{2}$

比較 $s^2$ 項: $E+F=10 \quad \Rightarrow F=\frac{41}{4}$

$$\therefore \quad Y(s)=\frac{10s^2+11s}{(s+1)(s-1)^2}=\frac{-\frac{1}{4}}{(s+1)}+\frac{\frac{41}{4}}{(s-1)}+\frac{\frac{21}{2}}{(s-1)^2}$$

由 $X(s)=\frac{-5}{s}+\frac{\frac{1}{4}}{s+1}+\frac{\frac{19}{4}}{s-1}+\frac{-\frac{7}{2}}{(s-1)^2} \quad \Rightarrow x(t)=-5+\frac{1}{4}e^{-t}+\frac{19}{4}e^{t}-\frac{7}{2}te^{t}$

$$Y(s)=\frac{-\frac{1}{4}}{(s+1)}+\frac{\frac{41}{4}}{(s-1)}+\frac{\frac{21}{2}}{(s-1)^2} \quad \Rightarrow y(t)=-\frac{1}{4}e^{-t}+\frac{41}{4}e^{t}+\frac{21}{2}te^{t}$$