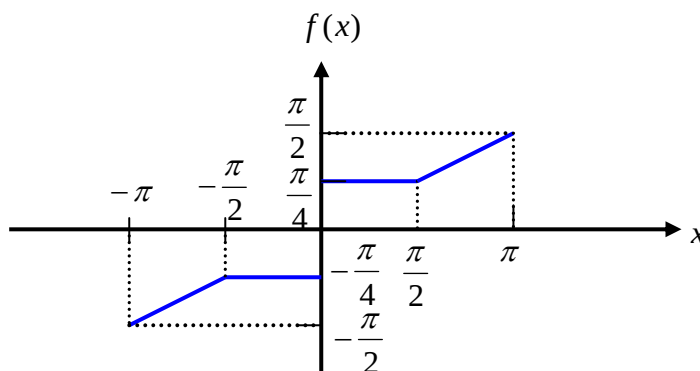


系級：_____ 學號：_____ 姓名：_____

1. 已知 $f(t) = \begin{cases} 0, & -\pi < t < 0 \\ \sin 2t, & 0 < t < \pi \end{cases}$

- (1) 試將 $f(t)$ 展成傅立葉級數。(8%) (2) 試將 $f(t)$ 展成傅立葉積分。(8%)
2. 給一週期函數如下圖所示，已知其週期為 2π ，試求此函數 $f(x)$ 之傅立葉級數，並列出前三個係數($a_0, a_1, a_2, a_3, b_1, b_2, b_3$)之值為何?(12%)



3. 給一函數 $f(x) = -2x^2$ 其中 $-1 < x < 1$ 並且又有 $f(x+2n) = f(x)$ ， n 為正整數

- (1) 請畫出函數 $f(x)$ 之圖形。(2%)
- (2) 試問此函數為奇函數或是偶函數?(2%) 週期 $T = ?$ (2%)
- (3) 試求 $f(x)$ 的傅立葉級數展開。(6%)
- (4) 試問： $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \dots = ?$ (5%)
- (5) 試問： $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots = ?$ (5%)

4. (1) 試求函數 $f(t) = \begin{cases} 0, & -\infty < t \leq -1 \\ 1+t, & -1 < t \leq 0 \\ 1-t, & 0 < t \leq 1 \\ 0, & 1 < t < \infty \end{cases}$ 的傅立葉積分表示式。(8%)

(2) 試問 $\int_0^\infty \frac{1 - \cos \omega}{\omega^2} d\omega = ?$ (5%)

5. 已知函數 $f(x) = e^{-|x|}$ 與 $g(x) = H(x+1) - H(x-1)$ ，其中 $H(x)$ 為單位步階函數，其定義為 $H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$

數，其定義為 $H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$

(1) 試求： $f(x)$ 之傅立葉轉換 $F(\omega) = ?$ (5%)

(2) 試求： $g(x)$ 之傅立葉轉換 $G(\omega) = ?$ (5%)

(3) 試求： $\int_{-\infty}^\infty \frac{\sin^2 \omega}{\omega^2} d\omega = ?$ (5%)

(4) 取 $q(x) = f(x) * g(x)$ ，試問： $q(x) = ?$ (5%)

(hint：分 $x < -1$ ， $-1 < x < 1$ ， $1 < x$ 討論)

(5) $Q(\omega) = \mathcal{F}[q(x)] = ?$ (5%)

6. (1) 試求 $f(x) = e^{-ax}u(x)$ 之傅立葉轉換 $F(\omega)$ ，其中 $a > 0$ 。(6%)

(2) 試求 $H(\omega) = \frac{e^{-4\omega i}}{3 + \omega i}$ 之傅立葉反轉換 $h(x)$ 。(6%)

傅立葉級數展開

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2n\pi x}{T} + b_n \sin \frac{2n\pi x}{T} \right)$$

$$a_0 = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) dx, \quad a_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) \cos \frac{2n\pi x}{T} dx, \quad b_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) \sin \frac{2n\pi x}{T} dx$$

傅立葉級數之 Parseval 恆等式：
$$\frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f^2(x) dx = a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

傅立葉積分：
$$f(x) = \int_0^{\infty} [A(\omega) \cos \omega x + B(\omega) \sin \omega x] d\omega$$

$$\text{其中 } A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \cos \omega x dx, \quad B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x) \sin \omega x dx$$

傅立葉複數形式級數展開

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n x}, \quad \text{其中 } \omega_n = \frac{2n\pi}{T}, \quad c_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) e^{-i\omega_n x} dx$$

傅立葉轉換：
$$F(\omega) = \mathcal{F}[f(x)] = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

傅立葉反轉換：
$$f(x) = \mathcal{F}^{-1}[F(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x} d\omega$$

傅立葉轉換的 Parseval 恆等式：
$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

Convolution：
$$f * g = \int_{-\infty}^{\infty} f(t - \tau) g(\tau) d\tau \Rightarrow \mathcal{F}[f(t) * g(t)] = F(\omega) \cdot G(\omega)$$

$$\mathcal{F}[f'(t)] = i\omega F(\omega) \Rightarrow \mathcal{F}[f^{(n)}(t)] = (i\omega)^n F(\omega)$$

$$\mathcal{F}[t^n f(t)] = i^n \frac{d^n}{d\omega^n} F(\omega)$$

$$\int_a^b f(x) \delta(x - x_0) dx = f(x_0) \quad \text{其中 } a < x_0 < b$$

$$\text{尤拉公式: } \cos x = \frac{1}{2}(e^{ix} + e^{-ix}), \quad \sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$$

$$\text{Scaling: } \mathcal{F}[f(at)] = \frac{1}{a} F\left(\frac{\omega}{a}\right)$$

$$\text{Time shifting: } \mathcal{F}[f(t - T)] = e^{-i\omega T} F(\omega)$$

$$\text{Frequency shifting: } \mathcal{F}[e^{i\omega_0 t} f(t)] = F(\omega - \omega_0)$$

參考解答:

1. 已知 $f(t) = \begin{cases} 0, & -\pi < t < 0 \\ \sin 2t, & 0 < t < \pi \end{cases}$ (111 成大土木)

(1) 試將 $f(t)$ 展成傅立葉級數。(8%)

(2) 試將 $f(t)$ 展成傅立葉積分。(8%)

(1) 將 $f(t)$ 展成傅立葉級數，週期為 $T = 2\pi$

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{2n\pi t}{T} + \sum_{n=1}^{\infty} b_n \sin \frac{2n\pi t}{T} = a_0 + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$$

$$a_0 = \frac{1}{T} \int_0^T f(t) dt = \frac{1}{2\pi} \int_0^{2\pi} f(t) dt = \frac{1}{2\pi} \int_0^{\pi} \sin 2t dt = 0$$

$$\begin{aligned} a_n &= \frac{2}{T} \int_0^T f(t) \cos \frac{2n\pi t}{T} dt = \frac{1}{\pi} \int_0^{\pi} \sin 2t \cdot \cos nt dt \\ &= \frac{1}{2\pi} \int_0^{\pi} [\sin(2+n)t + \sin(2-n)t] dt \\ &= -\frac{1}{2\pi} \left[\frac{1}{2+n} \cos(2+n)t \Big|_0^{\pi} + \frac{1}{2-n} \cos(2-n)t \Big|_0^{\pi} \right] \\ &= \begin{cases} \frac{2[1 - (-1)^n]}{\pi(4 - n^2)}, & n \neq 2 \\ 0, & n = 2 \end{cases} \end{aligned}$$

$$b_n = \frac{2}{T} \int_0^T f(t) \sin \frac{2n\pi t}{T} dt = \frac{1}{\pi} \int_0^{\pi} \sin 2t \cdot \sin nt dt = \begin{cases} 0, & n \neq 2 \\ \frac{1}{2}, & n = 2 \end{cases}$$

$$\therefore f(t) = -\frac{4}{\pi} \left(\frac{1}{1 \cdot 3} \cos t + \frac{1}{1 \cdot 5} \cos 3t + \frac{1}{3 \cdot 7} \cos 5t + \cdots \right) + \frac{1}{2} \sin 2t$$

(2) 將 $f(t)$ 展成傅立葉積分

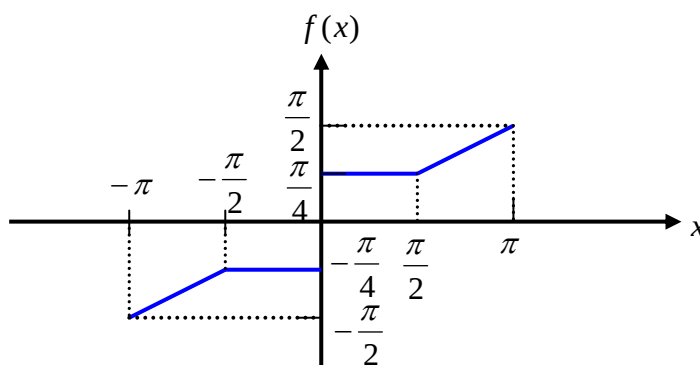
$$f(t) = \int_0^{\infty} [A(\omega) \cos \omega t + B(\omega) \sin \omega t] d\omega$$

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos \omega t dt = \frac{1}{\pi} \int_0^{\pi} \sin 2t \cdot \cos \omega t dt = \begin{cases} \frac{2[1 - \cos \omega\pi]}{\pi(4 - \omega^2)}, & \omega \neq 2 \\ 0, & \omega = 2 \end{cases}$$

$$B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin \omega t dt = \frac{1}{\pi} \int_0^{\pi} \sin 2t \cdot \sin \omega t dt = \begin{cases} \frac{2 \sin \omega\pi}{\pi(4 - \omega^2)}, & \omega \neq 2 \\ \frac{1}{2}, & \omega = 2 \end{cases}$$

2. 給一週期函數如下圖所示，已知其週期為 2π ，試求此函數 $f(x)$ 之傅立葉級數，並列出前三個係數($a_0, a_1, a_2, a_3, b_1, b_2, b_3$)之值為何? (12%)

(110 陽明交大土木)



由圖可知此為奇函數，週期為 $T = 2\pi$ 且 $a_0 = a_n = 0$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{2n\pi x}{T} + \sum_{n=1}^{\infty} b_n \sin \frac{2n\pi x}{T} = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) \sin \frac{2n\pi x}{T} dx = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \sin nx dx = \frac{2}{\pi} \int_0^{\pi} f(x) \cdot \sin nx dx$$

$$= \frac{2}{\pi} \left[\int_0^{\frac{\pi}{2}} \frac{\pi}{4} \sin nx dx + \int_{\frac{\pi}{2}}^{\pi} \frac{x}{2} \cdot \sin nx dx \right]$$

$$= \frac{2}{\pi} \left[-\frac{\pi}{4n} \cos nx \Big|_0^{\frac{\pi}{2}} + \frac{1}{2} \left(-\frac{x}{n} \cos nx + \frac{1}{n^2} \sin nx \right) \Big|_{\frac{\pi}{2}}^{\pi} \right]$$

$$= \frac{1}{2n} (1 - \cos \frac{n\pi}{2}) - \frac{1}{n\pi} \left[\pi \cos n\pi - \frac{\pi}{2} \cos \frac{n\pi}{2} \right] + \frac{1}{n^2\pi} (\sin n\pi - \sin \frac{n\pi}{2})$$

$$= \frac{1}{2n} (1 - \cos \frac{n\pi}{2}) - \frac{1}{n} [(-1)^n - \frac{1}{2} \cos \frac{n\pi}{2}] - \frac{1}{n^2\pi} \sin \frac{n\pi}{2}$$

$$= \frac{1 - 2(-1)^n}{2n} - \frac{1}{n^2\pi} \sin \frac{n\pi}{2}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \left[\frac{1 - 2(-1)^n}{2n} - \frac{1}{n^2\pi} \sin \frac{n\pi}{2} \right] \sin nx$$

$$\therefore a_0 = a_1 = a_2 = a_3 = 0, \quad b_1 = \frac{3}{2} - \frac{1}{\pi}, \quad b_2 = -\frac{1}{4}, \quad b_3 = \frac{1}{2} + \frac{1}{9\pi}$$

3. 給一函數 $f(x) = -2x^2$ 其中 $-1 < x < 1$ 並且又有 $f(x+2n) = f(x)$ ， n 為正整數

(1) 請畫出函數 $f(x)$ 之圖形。(2%)

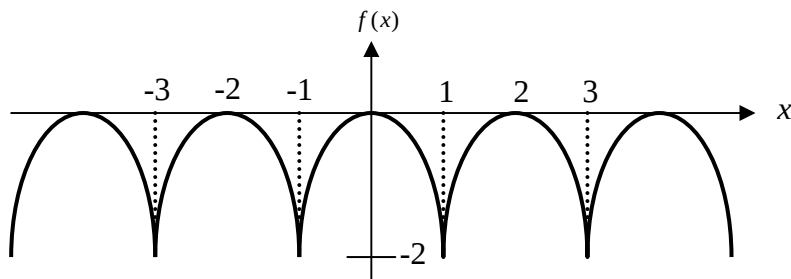
(2) 試問此函數為奇函數或是偶函數?(2%) 週期 $T = ?$ (2%)

(3) 試求 $f(x)$ 的傅立葉級數展開。(6%)

(4) 試問: $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \dots = ?$ (5%)

(5) 試問: $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots = ?$ (5%)

(1)



(2) 偶函數， $T = 2$

$$(3) \quad f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{2n\pi x}{T} + \sum_{n=1}^{\infty} b_n \sin \frac{2n\pi x}{T} = a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x$$

$$a_0 = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) dx = \int_0^1 (-2x^2) dx = -\frac{2}{3}$$

$$a_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) \cos \frac{2n\pi x}{T} dx = \int_{-1}^1 f(x) \cdot \cos n\pi x dx$$

$$= 2 \int_0^1 (-2x^2) \cdot \cos n\pi x dx = \frac{8(-1)^{n+1}}{n^2 \pi^2}$$

$$\therefore f(x) = -\frac{2}{3} + \sum_{n=1}^{\infty} \frac{8(-1)^{n+1}}{n^2 \pi^2} \cos n\pi x$$

$$(4) \quad \text{當 } x=0 \text{ 時, } 0 = -\frac{2}{3} + \sum_{n=1}^{\infty} \frac{8(-1)^{n+1}}{n^2 \pi^2}$$

$$\Rightarrow 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

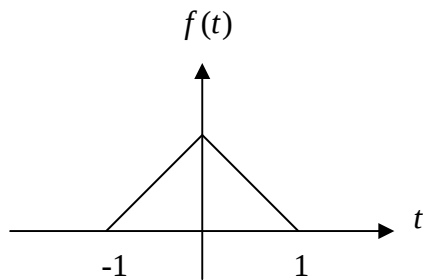
$$(5) \quad \text{當 } x=1 \text{ 時, } -2 = -\frac{2}{3} - \sum_{n=1}^{\infty} \frac{8}{n^2 \pi^2}$$

$$\Rightarrow 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

4. (1) 試求函數 $f(t) = \begin{cases} 0, & -\infty < t \leq -1 \\ 1+t, & -1 < t \leq 0 \\ 1-t, & 0 < t \leq 1 \\ 0, & 1 < t < \infty \end{cases}$ 的傅立葉積分表示式。(8%)

(2) 試問 $\int_0^\infty \frac{1-\cos \omega}{\omega^2} d\omega = ?$ (5%)

(1)



$\therefore$ 由圖可知此為偶函數

$$\therefore f(t) = \int_0^\infty A(\omega) \cos \omega t d\omega \quad (B(\omega) = 0)$$

$$\begin{aligned} A(\omega) &= \frac{1}{\pi} \int_{-\infty}^\infty f(t) \cos \omega t dt = \frac{2}{\pi} \int_0^1 (1-t) \cos \omega t dt \\ &= \frac{2}{\pi \omega^2} (1 - \cos \omega) \end{aligned}$$

$$\therefore f(t) = \frac{2}{\pi} \int_0^\infty \frac{(1 - \cos \omega) \cos \omega t}{\omega^2} d\omega$$

$$(2) \quad f(0) = 1 = \frac{2}{\pi} \int_0^\infty \frac{1 - \cos \omega}{\omega^2} d\omega \Rightarrow \int_0^\infty \frac{1 - \cos \omega}{\omega^2} d\omega = \frac{\pi}{2}$$

5. 已知函數 $f(x) = e^{-|x|}$ 與 $g(x) = H(x+1) - H(x-1)$ ，其中 $H(x)$ 為單位步階函

數，其定義為 $H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$

(1) 試求： $f(x)$ 之傅立葉轉換 $F(\omega) = ?$ (5%)

(2) 試求： $g(x)$ 之傅立葉轉換 $G(\omega) = ?$ (5%)

(3) 試求： $\int_{-\infty}^{\infty} \frac{\sin^2 \omega}{\omega^2} d\omega = ?$ (5%)

(4) 取 $q(x) = f(x) * g(x)$ ，試問： $q(x) = ?$ (5%)

(hint：分 $x < -1$ ， $-1 < x < 1$ ， $1 < x$ 討論)

(5) $Q(\omega) = \mathcal{F}[q(x)] = ?$ (5%)

(1) $\because f(x) = e^{-|x|}$ 為偶函數

$$\begin{aligned} \mathcal{F}[f(x)] &= F(\omega) = \int_{-\infty}^{\infty} e^{-|x|} e^{-i\omega x} dx \\ &= 2 \int_0^{\infty} e^{-x} \cos \omega x dx \\ \therefore &= \frac{2}{1+\omega^2} (-e^{-x} \cos \omega x + \omega e^{-x} \sin \omega x) \Big|_0^{\infty} \\ &= \frac{2}{1+\omega^2} \end{aligned}$$

(2) $\because g(x) = H(x+1) - H(x-1)$ 為偶函數

$$\begin{aligned} \mathcal{F}[g(x)] &= G(\omega) = \int_{-\infty}^{\infty} [H(x+1) - H(x-1)] e^{-i\omega x} dx \\ &= 2 \int_0^1 \cos \omega x dx \\ &= \frac{2}{\omega} \sin \omega \end{aligned}$$

(3) 傅立葉轉換的 Parseval 恆等式可得

$$\begin{aligned} \int_{-\infty}^{\infty} |H(x+1) - H(x-1)|^2 dx &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{2}{\omega} \sin \omega \right|^2 d\omega \\ \Rightarrow \int_{-1}^1 dx &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{4}{\omega^2} \sin^2 \omega d\omega \\ \Rightarrow \int_{-\infty}^{\infty} \frac{\sin^2 \omega}{\omega^2} d\omega &= \pi \end{aligned}$$

$$\begin{aligned}
 (4) \quad q(x) &= f * g = \int_{-\infty}^{\infty} f(x-\tau)g(\tau)d\tau \\
 &= \int_{-\infty}^{\infty} e^{-|x-\tau|}[H(\tau+1)-H(\tau-1)]d\tau \\
 &= \int_{-1}^1 e^{-|x-\tau|} d\tau
 \end{aligned}$$

$$\text{當 } x > 1 \text{ 時 } q(x) = \int_{-1}^1 e^{-|x-\tau|} d\tau = \int_{-1}^1 e^{-x+\tau} d\tau = e^{-x+1} - e^{-x-1}$$

$$\text{當 } x < -1 \text{ 時 } q(x) = \int_{-1}^1 e^{-|x-\tau|} d\tau = \int_{-1}^1 e^{x-\tau} d\tau = -e^{x-1} + e^{x+1}$$

當 $-1 < x < 1$ 時

$$\begin{aligned}
 q(x) &= \int_{-1}^1 e^{-|x-\tau|} d\tau = \int_{-1}^x e^{-x+\tau} d\tau + \int_x^1 e^{x-\tau} d\tau \\
 &= (1 - e^{-x-1}) - (e^{x-1} - 1) \\
 &= 2 - e^{-x-1} - e^{x-1}
 \end{aligned}$$

$$(5) \quad Q(\omega) = \mathcal{F}[q(x)] = F(\omega)G(\omega) = \frac{2}{1+\omega^2} \cdot \frac{2}{\omega} \sin \omega x = \frac{4}{\omega(1+\omega^2)} \sin \omega$$

6. (1) 試求 $f(x) = e^{-ax}u(x)$ 之傅立葉轉換 $F(\omega)$ ，其中 $a > 0$ 。(6%)

(2) 試求 $H(\omega) = \frac{e^{-4\omega i}}{3+\omega i}$ 之傅立葉反轉換 $h(x)$ 。(6%)

$$(1) \quad F(\omega) = \mathcal{F}[f(x)] = \int_{-\infty}^{\infty} f(x)e^{-i\omega x} dx = \int_{-\infty}^{\infty} e^{-ax}u(x)e^{-i\omega x} dx = \int_0^{\infty} e^{-(a+i\omega)x} dx$$

$$= -\frac{1}{a+i\omega} e^{-(a+i\omega)x} \Big|_0^{\infty} = \frac{1}{a+i\omega}$$

(2) 由平移定理可知 $\mathcal{F}[f(x-T)] = e^{-i\omega T} F(\omega)$

$$\text{又由(2)之結果可知 } \mathcal{F}^{-1}\left[\frac{1}{3+\omega i}\right] = e^{-3x}u(x)$$

$$\therefore \mathcal{F}^{-1}[H(\omega)] = \mathcal{F}^{-1}\left[\frac{e^{-4\omega i}}{3+\omega i}\right] = e^{-3(x-4)}u(x-4)$$