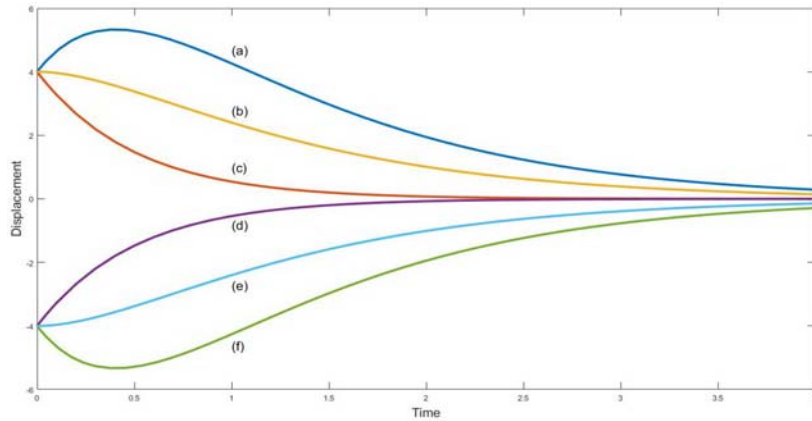


系級：_____ 學號：_____ 姓名：_____

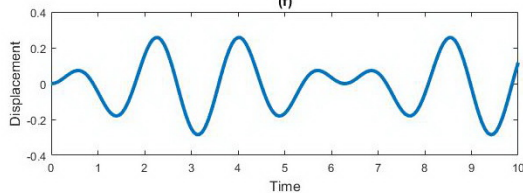
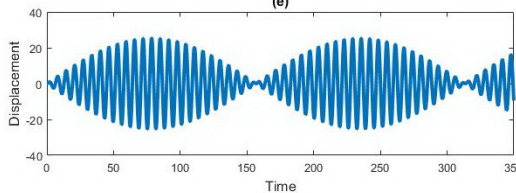
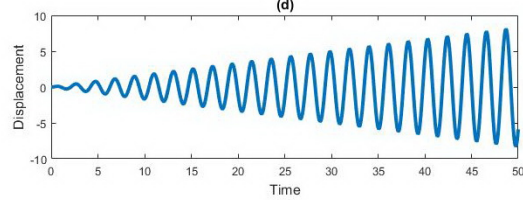
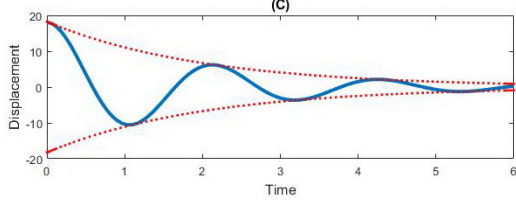
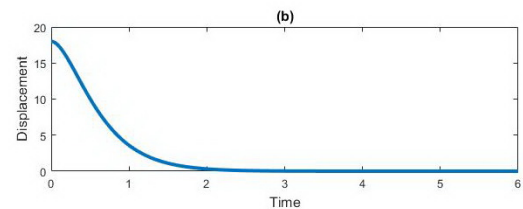
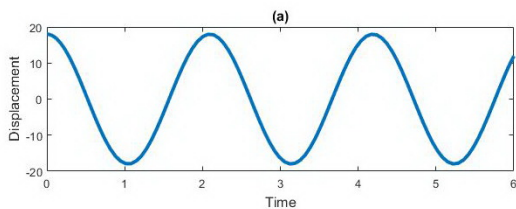
1. 給一單自由度振動系統，其控制方程式為 $m\ddot{y} + c\dot{y} + ky = 0$ ，已知相關參數 $m=1, c=3, k=2$ ，試問不同初始條件所對應之位移圖為何？(9%)

- (1) 初始條件： $y(0) = 4, \dot{y}(0) = 0$
- (2) 初始條件： $y(0) = -4, \dot{y}(0) = 8$
- (3) 初始條件： $y(0) = -4, \dot{y}(0) = -8$



2. 考慮下述三條微分方程式，試問其所對應之位移圖為何？(9%)

- (1) $y''(t) + y'(t) + 9y(t) = 0, y(0) = 18, \dot{y}(0) = 0$
- (2) $y''(t) + 9y(t) = \cos(3t), y(0) = 0, \dot{y}(0) = 0$
- (3) $y''(t) + 9y(t) = 0, y(0) = 18, \dot{y}(0) = 0$



3. 給一非齊次線性微分方程如下:

$$xy''(x) + ay'(x) + \frac{b}{x}y(x) = x^{-3}$$

已知此微分方程的兩個補解為 x^{-2} 與 $x^{-2} \ln x$

- (1) 試求常數 a 、 b 為何? (4%)
- (2) 以變數變換, 令 $t = \ln x$, 則 $y(x) = Y(t)$, 試求轉換後以 $Y(t)$ 表示的微分方程式。 (6%)
- (3) 試求轉換後微分方程的補解 $Y_h(t) = ?$ (4%)
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- (5) 試將 $Y(t)$ 轉換回 $y(x)$ 。 (4%)

4. 試求下述微分方程之通解

- (1) $y^{(6)} + 2y^{(4)} + y'' = 0$ (8%)
- (2) $y'' - 2y' + y = \frac{e^x}{x}$ (8%)
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- (4) $2(3x + 1)^2 y'' + 21(3x + 1)y' + 18y = 0$ (8%)

5. 已知微分方程式 $xy'' - (2x + 1)y' + (x + 1)y = (x^2 + x - 1)e^{2x}$

- (1) 已知 $y_1 = e^{ax}$ 為上述 ODE 之一補解, 試問 $a = ?$ (2%)
- (2) 試問: 另一補解與特解分別為何? (8%)

6. 已知單自由度振動系統其數學表示為 $m\ddot{y}(t) + c\dot{y}(t) + ky(t) = f(t)$, 若給定質量塊 $m = 2$, 阻尼係數 $c = 0$ 與彈簧常數 $k = 8$ 並且質量塊為靜止狀態即其初始條件 $y(0) = 0$ 與 $\dot{y}(0) = 0$, 給一外力為 $f(t) = 4 \cos \omega t$, 試問:

- (1) 此系統的自然振動頻率 $\omega_n = ?$ (2%)
- (2) 若 y_1, y_2 為其兩補解, 試求: $W(y_1, y_2) = ?$ (4%)
- (3) 當 $\omega = 2$, 其解為何? (4%) 此時系統的運動行為稱為什麼? (2%)
- (4) 當 $\omega = 1.99$, 其解為何? (4%) 此時系統的運動行為稱為什麼? (2%)

參考解答:

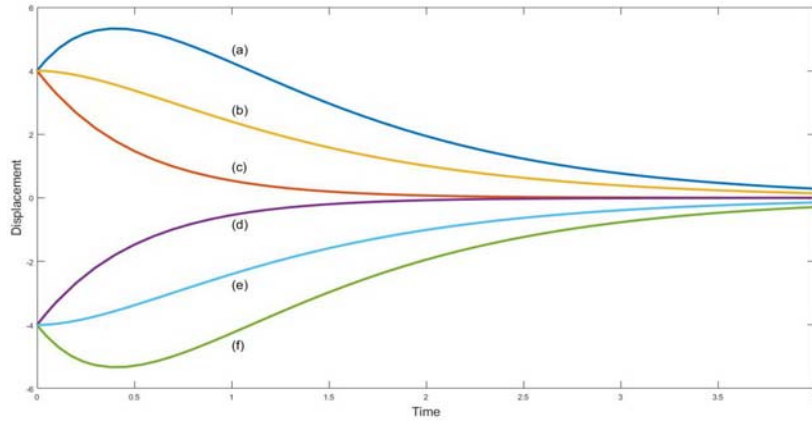
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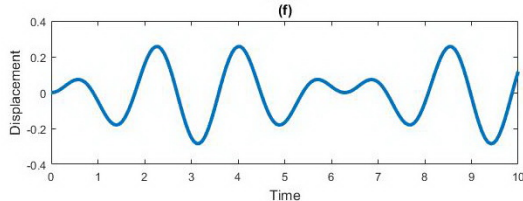
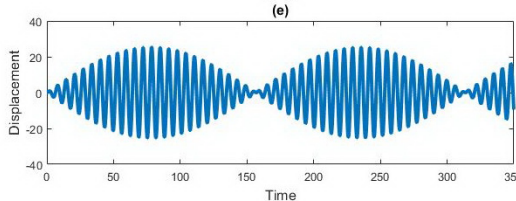
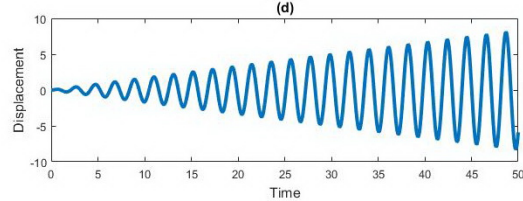
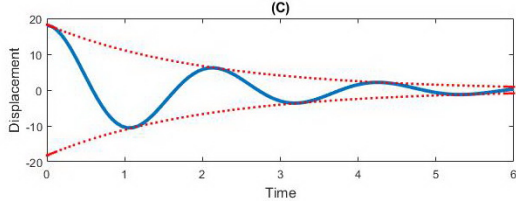
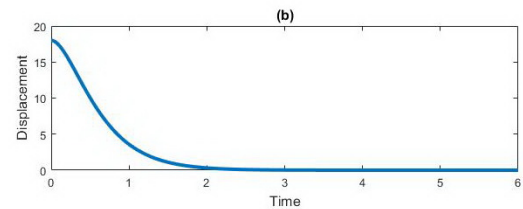
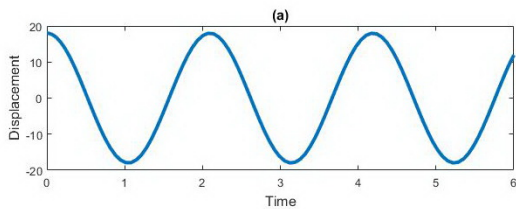
(1) b (2) d (3) f

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- (5) 試將 $Y(t)$ 轉換回 $y(x)$ 。 (4%)

$$(1) \quad xy''(x) + ay'(x) + \frac{b}{x}y(x) = x^{-3} \quad \Rightarrow \quad x^2y''(x) + axy'(x) + by(x) = x^{-2}$$

$\therefore a$ 、 b 為常數

\therefore 可知此為 Euler-Cauchy ODE

又 x^{-2} 與 $x^{-2} \ln x$ 為此微分方程的 2 個補解 ($m = -2$ 重根)

故可知此 ODE 之特徵方程為 $(m+2)^2 = 0$

$$\therefore m(m-1) + am + b = (m+2)^2$$

$$\Rightarrow m^2 + (a-1)m + b = m^2 + 4m + 4$$

比較係數後可得 $a = 5$ 與 $b = 4$

可知此 ODE 為 $x^2y''(x) + 5xy'(x) + 4y(x) = x^{-2}$

$$(2) \quad \text{令 } t = \ln x \Rightarrow x = e^t$$

$$\therefore y(x) = y(e^t) = Y(t)$$

$$y'(x) = \frac{dy(x)}{dx} = \frac{dY(t)}{dx} = \frac{dt}{dx} \cdot \frac{dY(t)}{dt} = \frac{1}{x} Y'(t)$$

$$y''(x) = \frac{dy'(x)}{dx} = \frac{d}{dx} \left(\frac{1}{x} Y'(t) \right) = -\frac{1}{x^2} Y'(t) + \frac{1}{x} \frac{dY'(t)}{dx} = -\frac{1}{x^2} Y'(t) + \frac{1}{x^2} Y''(t)$$

將 $y'(x)$ 與 $y''(x)$ 代回 ODE 可得

$$x^2 \left[-\frac{1}{x^2} Y'(t) + \frac{1}{x^2} Y''(t) \right] + 5x \cdot \frac{1}{x} Y'(t) + 4Y(t) = e^{-2t}$$

$$\Rightarrow Y''(t) + 4Y'(t) + 4Y(t) = e^{-2t}$$

(3) \therefore 此為常係數 ODE

\therefore 令 $y = e^{\lambda x}$ 代入 ODE 可得

$$(\lambda^2 + 4\lambda + 4)e^{\lambda x} = 0$$

$$\Rightarrow \lambda^2 + 4\lambda + 4 = 0$$

$$\Rightarrow \lambda = -2, -2 \quad (\text{重根})$$

$$\therefore Y_h = c_1 e^{-2t} + c_2 e^{-2t} \cdot t$$

(4) 由待定係數法，令 $Y_p = At^2e^{-2t}$ 代回 ODE 可得

$$Y_p' = 2Ate^{-2t} - 2At^2e^{-2t} = A(2t - 2t^2)e^{-2t}$$

$$Y_p'' = A(2 - 4t)e^{-2t} - 2A(2t - 2t^2)e^{-2t} = A(2 - 8t + 4t^2)e^{-2t}$$

$$A(2 - 8t + 4t^2)e^{-2t} + 4A(2t - 2t^2)e^{-2t} + 4At^2e^{-2t} = e^{-2t} \Rightarrow A = \frac{1}{2}$$

$$Y_p = \frac{1}{2}t^2e^{-2t}$$

(5) $Y(t) = Y_h(t) + Y_p(t) = c_1e^{-2t} + c_2e^{-2t} \cdot t + \frac{1}{2}t^2e^{-2t}$

$$\Rightarrow y(x) = c_1x^{-2} + c_2x^{-2} \cdot \ln x + \frac{1}{2}x^{-2} \cdot (\ln x)^2$$

4. 試求下述微分方程之通解

(1) $y^{(6)} + 2y^{(4)} + y'' = 0$ (8%)

(2) $y''' - 2y' + y = \frac{e^x}{x}$ (8%)

(3) $(1 - x^2)y'' - 4xy' - 2y = x$ (8%)

(4) $2(3x+1)^2 y'' + 21(3x+1)y' + 18y = 0$ (8%)

(1) 令 $y = e^{\lambda x}$ 代入 ODE 可得

$$(\lambda^6 + 2\lambda^4 + \lambda^2)e^{\lambda x} = 0$$

$$\Rightarrow \lambda^2(\lambda^2 + 1)^2 = 0$$

$$\Rightarrow \lambda = 0, 0, \pm i, \pm i$$

$$\therefore y = C_1 + C_2x + (C_3 \cos x + C_4 \sin x) + x(C_5 \cos x + C_6 \sin x)$$

(2) 求補解

\therefore 此為常係數 ODE

\therefore 令 $y = e^{\lambda x}$ 代入 ODE 可得

$$(\lambda^2 - 2\lambda + 1)e^{\lambda x} = 0$$

$$\Rightarrow (\lambda - 1)^2 = 0$$

$$\Rightarrow \lambda = 1, 1$$

$$\therefore y_h = c_1e^x + c_2xe^x$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x}$$

令其特解 $y_p(x) = u_1y_1 + u_2y_2 = u_1e^x + u_2xe^x$ 代回 ODE 後可得

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{W} = \frac{\begin{vmatrix} 0 & xe^x \\ \frac{e^x}{x} & e^x + xe^x \end{vmatrix}}{e^{2x}} = -1 \Rightarrow u_1 = -x$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & \frac{e^x}{x} \end{vmatrix}}{W} = \frac{\begin{vmatrix} e^x & 0 \\ e^x & \frac{e^x}{x} \end{vmatrix}}{e^{2x}} = \frac{1}{x} \Rightarrow u_2 = \ln|x|$$

$$\therefore y_p(x) = u_1 e^x + u_2 \cdot x e^x = -x e^x + \ln|x| \cdot x e^x = x e^x (-1 + \ln|x|)$$

通解 $y = y_h(x) + y_p(x) = C_1 e^x + C_2 x e^x + x e^x \cdot \ln|x|$

(3) $(1-x^2)y'' - 4xy' - 2y = x$

令 $a_2 = (1-x^2)$, $a_1 = -4x$, $a_0 = -2$

由判斷式: $a_2'' - a_1' + a_0 = 0$ 可知此為正合式

$$(1-x^2)y'' - 4xy' - 2y = \frac{d}{dx}[b_1(x)y' + b_0(x)y]$$

$$\Rightarrow b_1(x)y'' + [b_1'(x) + b_0(x)] + b_0'(x)y = (1-x^2)y'' - 4xy' - 2y$$

$$\Rightarrow b_1 = 1-x^2, \quad b_0 = -2x$$

$$\therefore (1-x^2)y'' - 4xy' - 2y = \frac{d}{dx}[(1-x^2)y' - 2xy] = x$$

$$\Rightarrow (1-x^2)y' - 2xy = \frac{1}{2}x^2 + C_1 \quad \text{此為一階線性 ODE}$$

$$\Rightarrow y' - \frac{2x}{1-x^2}y = \frac{1}{2} \frac{x^2}{1-x^2} + \frac{C_1}{1-x^2}$$

積分因子: $\mu = e^{\int \frac{-2x}{1-x^2} dx} = e^{\int \frac{1}{1-x^2} d(1-x^2)} = e^{\ln|1-x^2|} = 1-x^2$

同乘積分因子: $(1-x^2)y' - 2xy = \frac{1}{2}x^2 + C_1$

$$\Rightarrow \frac{d}{dx}[(1-x^2)y] = \frac{1}{2}x^2 + C_1$$

$$\Rightarrow (1-x^2)y = \frac{1}{6}x^3 + C_1x + C_2$$

$$\Rightarrow y = \frac{x^3}{6(1-x^2)} + \frac{C_1x}{1-x^2} + \frac{C_2}{1-x^2}$$

$$(4) 2(3x+1)^2 y'' + 21(3x+1)y' + 18y = 0$$

$$\text{令 } t = 3x+1 \Rightarrow \frac{dy(x)}{dx} = \frac{dt}{dx} \cdot \frac{dY(t)}{dt} = 3Y'(t)$$

$$\Rightarrow \frac{d^2 y(x)}{dx^2} = \frac{d}{dx} \left(\frac{dy(x)}{dx} \right) = 3 \frac{dt}{dx} \cdot \frac{dY'(t)}{dt} = 9Y''(t)$$

$$2(3x+1)^2 y'' + 21(3x+1)y' + 18y = 0 \Rightarrow 18t^2 Y'' + 63tY' + 18Y = 0$$

$$\text{令 } Y = t^m \Rightarrow [18m(m-1) + 63m + 18]t^m = 0$$

$$\Rightarrow 18m^2 + 45m + 18 = 0$$

$$\Rightarrow (2m+1)(9m+18) = 0$$

$$\Rightarrow m = -\frac{1}{2} \text{ or } -2$$

$$\therefore Y(t) = C_1 t^{-\frac{1}{2}} + C_2 t^{-2}$$

$$\Rightarrow y(x) = C_1 (3x+1)^{-\frac{1}{2}} + C_2 (3x+1)^{-2}$$

5. 已知微分方程式 $xy'' - (2x+1)y' + (x+1)y = (x^2 + x - 1)e^{2x}$

(1) 已知 $y_1 = e^{ax}$ 為上述 ODE 之一補解，試問 $a = ?$ (2%)

(2) 試問：另一補解與特解分別為何? (8%)

$$(1) xy'' - (2x+1)y' + (x+1)y = 0$$

令 $y_1 = e^{ax}$ 代入 ODE 可得

$$a^2 x e^{ax} - a(2x+1)e^{ax} + (x+1)e^{ax} = 0$$

$$\Rightarrow a^2 x - 2ax - a + x + 1 = 0$$

$$\Rightarrow a = 1$$

故可得一補解為 $y_1 = e^x$

$$(2) \text{ 令另一補解 } y_2 = vy_1 \Rightarrow y_2' = v'y_1 + vy_1'$$

$$\Rightarrow y_2'' = v''y_1 + 2v'y_1' + vy_1''$$

$$\text{代入 ODE: } xy'' - (2x+1)y' + (x+1)y = (x^2 + x - 1)e^{2x}$$

$$\text{可得 } x(v''y_1 + 2v'y_1' + vy_1'') - (2x+1)(v'y_1 + vy_1') + (x+1)vy_1 = (x^2 + x - 1)e^{2x}$$

$$\Rightarrow x(v''y_1 + 2v'y_1') - (2x+1)v'y_1 = (x^2 + x - 1)e^{2x}$$

$$\Rightarrow (xy_1)v'' + (2xy_1' - 2xy_1 - y_1)v' = (x^2 + x - 1)e^{2x}$$

$$\text{令 } z = v' \Rightarrow z' + \frac{2xy_1' - 2xy_1 - y_1}{xy_1} z = \frac{(x^2 + x - 1)e^{2x}}{xy_1}$$

$$\Rightarrow z' - \frac{1}{x} z = (x+1 - \frac{1}{x})e^x \longrightarrow \text{一階線性 ODE}$$

$$\text{積分因子為 } \mu = e^{\int p(x)dx} = e^{-\int \frac{1}{x} dx} = e^{-\ln x} = \frac{1}{x}$$

$$\begin{aligned}
&\Rightarrow \frac{1}{x}z' - \frac{1}{x^2}z = \left(1 + \frac{1}{x} - \frac{1}{x^2}\right)e^x \\
&\Rightarrow \frac{d}{dx}\left(\frac{1}{x}z\right) = \left(1 + \frac{1}{x} - \frac{1}{x^2}\right)e^x \\
&\Rightarrow \int d\left(\frac{1}{x}z\right) = \int \left(1 + \frac{1}{x} - \frac{1}{x^2}\right)e^x dx \\
&\Rightarrow \frac{1}{x}z = \left(e^x + \frac{e^x}{x}\right) + C_1 \\
&\Rightarrow z = (xe^x + e^x) + C_1x \\
&\Rightarrow v' = (xe^x + e^x) + C_1x \\
&\Rightarrow v = xe^x + \frac{C_1}{2}x^2 + C_2 \\
&y_2 = vy_1 = \left(xe^x + \frac{C_1}{2}x^2 + C_2\right)e^x = xe^{2x} + C_3x^2e^x + C_2e^x
\end{aligned}$$

\therefore 另一補解為 x^2e^x ，特解為 xe^{2x}

6. 已知單自由度振動系統其數學表示為 $m\ddot{y}(t) + c\dot{y}(t) + ky(t) = f(t)$ ，若給定質量塊 $m=2$ ，阻尼係數 $c=0$ 與彈簧常數 $k=8$ 並且質量塊為靜止狀態即其初始條件 $y(0)=0$ 與 $\dot{y}(0)=0$ ，給一外力為 $f(t) = 4\cos\omega t$ ，試問：

- (1) 此系統的自然振動頻率 $\omega_n = ?$ (2%)
- (2) 若 y_1, y_2 為其兩補解，試求： $W(y_1, y_2) = ?$ (4%)
- (3) 當 $\omega = 2$ ，其解為何？ (4%) 此時系統的運動行為稱為什麼？ (2%)
- (4) 當 $\omega = 1.99$ ，其解為何？ (4%) 此時系統的運動行為稱為什麼？ (2%)

(1) $m\ddot{y}(t) + c\dot{y}(t) + ky(t) = f(t)$ 又 $m=2, c=0, k=8$ 與 $f(t) = 4\cos\omega t$
 $\therefore 2\ddot{y}(t) + 8y(t) = 4\cos\omega t \Rightarrow \ddot{y}(t) + 4y(t) = 2\cos\omega t$

$$\omega_n = \sqrt{\frac{k}{m}} = 2$$

(2) 令 $y_1 = \cos 2t, y_2 = \sin 2t$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix} = 2$$

(3) 當 $\omega = \omega_n = 2$ 時，外力對系統產生共振行為
 $\ddot{y}(t) + 4y(t) = 2\cos 2t$

令 $y_p = t(A\cos 2t + B\sin 2t) = t \cdot y_h$

$$\dot{y}_p = y_h + t\dot{y}_h$$

$$\ddot{y}_p = \dot{y}_h + \dot{y}_h + t\ddot{y}_h = 2\dot{y}_h + t\ddot{y}_h \text{ 代回 ODE 可得}$$

$$\begin{aligned}
(2\dot{y}_h + t \cdot \ddot{y}_h) + t \cdot y_h &= 2 \cos 2t \\
\Rightarrow 2\dot{y}_h &= 2 \cos 2t \\
\Rightarrow (-2A \sin 2t + 2B \cos 2t) &= \cos 2t \\
\therefore A &= 0, \quad B = \frac{1}{2}
\end{aligned}$$

$$y(t) = y_h(t) + y_p(t) = C_1 \cos 2t + C_2 \sin 2t + \frac{1}{2}t \cdot \sin 2t$$

$$\text{又 } y(0) = 0 \Rightarrow C_1 = 0$$

$$\dot{y}(0) = 0 \Rightarrow C_2 = 0$$

$$\therefore y(t) = \frac{1}{2}t \cdot \sin 2t$$

(4) 當 $\omega = 1.99 \approx \omega_n$ 時，外力對系統產生拍擊行為

$$\ddot{y}(t) + 4y(t) = 2 \cos(1.99t)$$

$$\begin{aligned}
\Rightarrow y(t) &= \frac{2}{(\omega_n^2 - \omega^2)} (\cos \omega t - \cos \omega_n t) \\
&= \frac{4}{m(\omega_n^2 - \omega^2)} \sin \frac{\omega_n + \omega}{2} t \cdot \sin \frac{\omega_n - \omega}{2} t \\
&= \frac{2}{2^2 - 1.99^2} (\cos 1.99t - \cos 2t) \\
&= \frac{4}{2^2 - 1.99^2} \sin \frac{3.99}{2} t \cdot \sin \frac{0.01}{2} t
\end{aligned}$$