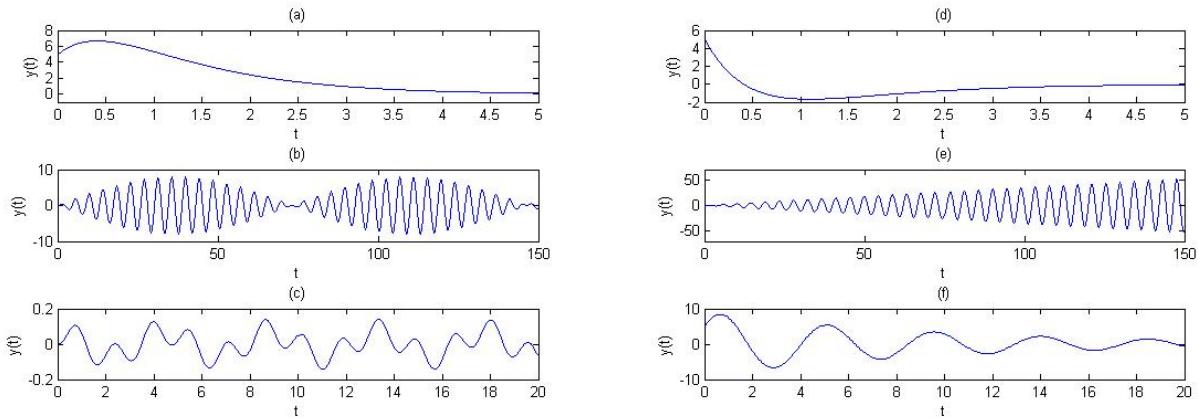


系級：_____ 學號：_____ 姓名：_____

1. 給一單自由度振動系統，其控制方程式為 $m\ddot{y} + c\dot{y} + ky = f(t)$ ，已知相關參數
如下表所示，試問其所對應之位移圖為何？(12%)

	系統參數	外力	初始條件	位移圖
(1)	$m=1, c=3, k=2$	$f(t)=0$	$y(0)=5, \dot{y}(0)=-20$	
(2)	$m=1, c=3, k=2$	$f(t)=0$	$y(0)=5, \dot{y}(0)=10$	
(3)	$m=1, c=0.2, k=2$	$f(t)=0$	$y(0)=5, \dot{y}(0)=10$	
(4)	$m=1, c=0, k=2$	$f(t)=\cos 4t$	$y(0)=0, \dot{y}(0)=0$	
(5)	$m=1, c=0, k=2$	$f(t)=\cos \sqrt{2}t$	$y(0)=0, \dot{y}(0)=0$	
(6)	$m=1, c=0, k=2$	$f(t)=\cos 1.5t$	$y(0)=0, \dot{y}(0)=0$	



2. 給一非齊次線性微分方程如下：

$$x^2 y''(x) + a x y'(x) + b y(x) = \ln x$$

已知此微分方程的兩個補解為 x 與 $x \ln x$

- (1) 試求常數 a 、 b 為何？(4%)
- (2) 以變數變換，令 $t = \ln x$ ，則 $y(x) = Y(t)$ ，試求轉換後以 $Y(t)$ 表示的微分方
程式。(6%)
- (3) 試求轉換後微分方程的補解 $Y_h(t) = ?$ (4%)
- (4) 試求轉換後微分方程的特解 $Y_p(t) = ?$ (4%)
- (5) 試將 $Y(t)$ 轉換回 $y(x)$ 。(4%)

3. 試求下述微分方程之通解

(1) $y''' + 3y'' + 3y' + y = e^{-x}$ (10%)

(2) $y'' + 2y' + y = \frac{1}{x^3 e^x}$ (10%)

(3) $(x+2)^2 y'' - (3x+6)y' + 4y = 3x+2$ 。 (10%)

(4) $xy'' + (x+2)y' + y = -2xe^{-x}$ 。 (10%)

4. 已知微分方程式 $(x^2 - 2x)y'' + 2(1-x)y' + 2y = 0$

(1) 試以 $y_1 = x^m$ 求一補解。 (3%)

(2) 試求另一補解。 (7%)

5. 紿一微分方程 $\ddot{y}(t) + \omega^2 y(t) = F \cos \gamma t$ ，其中 F 為常數且 $\gamma \neq \omega$

(1) 試求此微分方程之補解 $y_h(t) = ?$ (3%)

(2) 試求 Wronskian，即 $W(y_1, y_2) = ?$ (2%)

(3) 試求微分方程之通解 $y(t) = ?$ (5%)

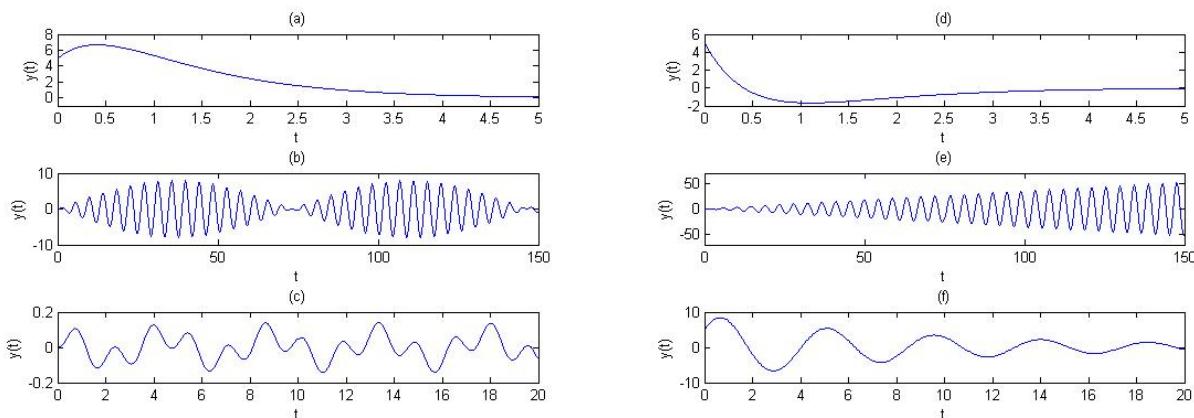
(4) 若給初始條件 $y(0) = 0$ 與 $\dot{y}(0) = 0$ ，則 $y(t) = ?$ (3%)

(5) 當 $\gamma = \omega$ 時，則 $\lim_{\gamma \rightarrow \omega} y(t) = ?$ (3%)

參考解答:

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(2)	$m=1, c=3, k=2$	$f(t)=0$	$y(0)=5, \dot{y}(0)=10$	(a)
(3)	$m=1, c=0.2, k=2$	$f(t)=0$	$y(0)=5, \dot{y}(0)=10$	(f)
(4)	$m=1, c=0, k=2$	$f(t)=\cos 4t$	$y(0)=0, \dot{y}(0)=0$	(c)
(5)	$m=1, c=0, k=2$	$f(t)=\cos \sqrt{2}t$	$y(0)=0, \dot{y}(0)=0$	(e)
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- (4) 試求轉換後微分方程的特解 $Y_p(t) = ?$ (4%)
- (5) 試將 $Y(t)$ 轉換回 $y(x)$ 。(4%)

(1) $\because a, b$ 為常數

\therefore 可知此為 Euler-Cauchy ODE

又 x 與 $x \ln x$ 為此微分方程的 2 個補解 ($m=1$ 重根)

故可知此 ODE 之特徵方程為 $(m-1)^2 = 0$

$$\therefore m(m-1) + am + b = (m-1)^2$$

$$\Rightarrow m^2 + (a-1)m + b = m^2 - 2m + 1$$

比較係數後可得 $a = -1$ 與 $b = 1$

可知此 ODE 為 $x^2 y''(x) - xy'(x) + y(x) = \ln x$

(2) 令 $t = \ln x \Rightarrow x = e^t$

$$\therefore y(x) = y(e^t) = Y(t)$$

$$y'(x) = \frac{dy(x)}{dx} = \frac{dY(t)}{dx} = \frac{dt}{dx} \cdot \frac{dY(t)}{dt} = \frac{1}{x} Y'(t)$$

$$y''(x) = \frac{dy'(x)}{dx} = \frac{d}{dx} \left(\frac{1}{x} Y'(t) \right) = -\frac{1}{x^2} Y'(t) + \frac{1}{x} \frac{dY'(t)}{dx} = -\frac{1}{x^2} Y'(t) + \frac{1}{x^2} Y''(t)$$

將 $y'(x)$ 與 $y''(x)$ 代回 ODE 可得

$$x^2 \left[-\frac{1}{x^2} Y'(t) + \frac{1}{x^2} Y''(t) \right] - x \cdot \frac{1}{x} Y'(t) + Y(t) = t$$

$$\Rightarrow Y''(t) - 2Y'(t) + Y(t) = t$$

(3) \because 此為常係數 ODE

\therefore 令 $y = e^{\lambda x}$ 代入 ODE 可得

$$(\lambda^2 - 2\lambda + 1)e^{\lambda x} = 0$$

$$\Rightarrow \lambda^2 - 2\lambda + 1 = 0$$

$$\Rightarrow \lambda = 1, 1 \text{ (重根)}$$

$$\therefore Y_h = c_1 e^t + c_2 e^t \cdot t$$

(4) 由待定係數法，令 $Y_p = At + B$ 代回 ODE 可得

$$-2A + At + B = t \Rightarrow A = 1 \text{ } \vee B = 2$$

$$\therefore Y_p = t + 2$$

$$(5) Y(t) = Y_h(t) + Y_p(t) = c_1 e^t + c_2 e^t \cdot t + t + 2$$

$$\Rightarrow y(x) = c_1 x + c_2 x \cdot \ln x + \ln x + 2$$

3. 試求下述微分方程之通解

(1) $y''' + 3y'' + 3y' + y = e^{-x}$ (10%)

(2) $y'' + 2y' + y = \frac{1}{x^3 e^x}$ (10%)

(3) $(x+2)^2 y'' - (3x+6)y' + 4y = 3x+2$ (10%)

(4) $xy'' + (x+2)y' + y = -2xe^{-x}$ (10%)

(1) 求補解

∴ 此為常係數 ODE

∴ 令 $y = e^{\lambda x}$ 代入 ODE 可得

$$(\lambda^3 + 3\lambda^2 + 3\lambda + 1)e^{\lambda x} = 0$$

$$\Rightarrow (\lambda + 1)^3 = 0$$

$$\Rightarrow \lambda = -1, -1, -1$$

$$\therefore y_h = e^{-x}(c_1 + c_2x + c_3x^2)$$

由待定係數法求特解

$$\text{令 } y_p = Ax^3e^{-x} \Rightarrow y'_p = A(-x^3 + 3x^2)e^{-x}$$

$$\Rightarrow y''_p = A(x^3 - 6x^2 + 6x)e^{-x}$$

$$\Rightarrow y'''_p = A(-x^3 + 9x^2 - 18x + 6)e^{-x} \quad \text{代入 ODE 可得}$$

$$A(-x^3 + 9x^2 - 18x + 6)e^{-x} + 3A(x^3 - 6x^2 + 6x)e^{-x} + 3A(-x^3 + 3x^2)e^{-x} \\ + Ax^3e^{-x} = e^{-x}$$

$$\Rightarrow A = \frac{1}{6}$$

$$\therefore y_p = \frac{1}{6}x^3e^{-x}$$

$$y = y_h + y_p = e^{-x}(c_1 + c_2x + c_3x^2) + \frac{1}{6}x^3e^{-x}$$

(2) 求補解

∴ 此為常係數 ODE

∴ 令 $y = e^{\lambda x}$ 代入 ODE 可得

$$(\lambda^2 + 2\lambda + 1)e^{\lambda x} = 0$$

$$\Rightarrow (\lambda + 1)^2 = 0$$

$$\Rightarrow \lambda = -1, -1$$

$$\therefore y_h = c_1e^{-x} + c_2xe^{-x}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{vmatrix} = e^{-2x}$$

令其特解 $y_p(x) = u_1 y_1 + u_2 y_2 = u_1 e^{-x} + u_2 x e^{-x}$ 代回 ODE 後可得

$$u'_1 = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y'_2 \end{vmatrix}}{W} = \frac{\begin{vmatrix} 0 & xe^{-x} \\ \frac{1}{x^3 e^x} & e^{-x} - xe^{-x} \end{vmatrix}}{e^{-2x}} = -\frac{1}{x^2} \Rightarrow u_1 = \frac{1}{x}$$

$$u'_2 = \frac{\begin{vmatrix} y_1 & 0 \\ y'_1 & \frac{1}{x^3 e^x} \end{vmatrix}}{W} = \frac{\begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & \frac{1}{x^3 e^x} \end{vmatrix}}{e^{-2x}} = \frac{1}{x^3} \Rightarrow u_2 = -\frac{1}{2x^2}$$

$$\therefore y_p(x) = u_1 e^{-x} + u_2 \cdot x e^{-x} = \frac{1}{x} e^{-x} - \frac{1}{2x^2} \cdot x e^{-x} = \frac{1}{2x} e^{-x}$$

$$\text{通解 } y = y_h(x) + y_p(x) = c_1 e^{-x} + c_2 x e^{-x} + \frac{1}{2x} e^{-x}$$

$$(3) (x+2)^2 y'' - (3x+6)y' + 4y = 3x+2$$

$$\begin{aligned} \text{令 } t = x+2 &\Rightarrow \frac{dy(x)}{dx} = \frac{dt}{dx} \cdot \frac{dY(t)}{dt} = Y'(t) \\ &\Rightarrow \frac{d^2 y(x)}{dx^2} = \frac{d}{dx} \left(\frac{dy(x)}{dx} \right) = \frac{dt}{dx} \cdot \frac{dY'(t)}{dt} = Y''(t) \end{aligned}$$

$$(x+2)^2 y'' - (3x+6)y' + 4y = 3x+2 \Rightarrow t^2 Y'' - 3t Y' + 4Y = 3t - 4$$

$$\begin{aligned} \text{令 } z = \ln t &\Rightarrow t = e^z \Rightarrow \frac{dY(t)}{dt} = \frac{dz}{dt} \cdot \frac{dG(z)}{dz} = \frac{1}{t} G'(z) \\ &\Rightarrow \frac{d^2 Y(t)}{dt^2} = -\frac{1}{t^2} G'(z) + \frac{1}{t} \frac{dz}{dt} \cdot \frac{dG'(z)}{dz} = \frac{1}{t^2} [G''(z) - G'(z)] \end{aligned}$$

$$t^2 Y'' - 3t Y' + 4Y = 3t - 4 \Rightarrow G'' - 4G' + 4G = 3e^z - 4$$

$$\text{令 } G = e^{\lambda z} \Rightarrow \lambda^2 - 4\lambda + 4 = 0 \Rightarrow \lambda = 2, 2$$

$$\therefore G_h(z) = c_1 e^{2z} + c_2 z e^{2z}$$

$$\text{令 } G_p = ae^z + b \text{ 代回 ODE 可得 } a = 3, b = -1$$

$$\therefore G(z) = G_h(z) + G_p(z) = c_1 e^{2z} + c_2 z e^{2z} + 3e^z - 1$$

$$\Rightarrow Y(t) = c_1 t^2 + c_2 t^2 \ln|t| + 3t - 1$$

$$\Rightarrow y(x) = c_1 (x+2)^2 + c_2 (x+2)^2 \cdot \ln|x+2| + 3x + 5$$

$$(4) xy'' + (x+2)y' + y = -2xe^{-x}$$

令 $a_2 = x$, $a_1 = x+2$, $a_0 = 1$

由判斷式: $a_2'' - a_1' + a_0 = 0$ 可知此為正合式

$$xy'' + (x+2)y' + y = \frac{d}{dx}[b_1(x)y' + b_0(x)y]$$

$$\Rightarrow b_1(x)y'' + [b_1'(x) + b_0(x)]y' + b_0'(x)y = xy'' + (x+2)y' + y$$

$$\Rightarrow b_1 = x, b_0 = x+1$$

$$\therefore xy'' + (x+2)y' + y = \frac{d}{dx}[xy' + (x+1)y] = -2xe^{-x}$$

$\Rightarrow xy' + (x+1)y = 2xe^{-x} + 2e^{-x} + c_1$ 此為一階線性 ODE

$$\Rightarrow y' + \frac{x+1}{x}y = 2e^{-x} + \frac{2}{x}e^{-x} + c_1 \frac{1}{x}$$

$$\text{積分因子: } \mu = e^{\int \frac{x+1}{x} dx} = e^{(x+\ln x)} = xe^x$$

$$\text{同乘積分因子: } xe^x y' + e^x (x+1)y = 2x + 2 + c_1 e^x$$

$$\Rightarrow \frac{d}{dx}(xe^x y) = 2x + 2 + c_1 e^x$$

$$\Rightarrow xe^x y = x^2 + 2x + c_1 e^x + c_2$$

$$\Rightarrow y = xe^{-x} + 2e^{-x} + c_1 \frac{1}{x} + c_2 \frac{e^{-x}}{x}$$

4. 已知微分方程式 $(x^2 - 2x)y'' + 2(1-x)y' + 2y = 0$

(1) 試以 $y_1 = x^m$ 求一補解。 (3%)

(2) 試求另一補解。 (7%)

(1) $(x^2 - 2x)y'' + 2(1-x)y' + 2y = 0$

令 $y_1 = x^m$ 代入 ODE 可得

$$m(m-1)(x^2 - 2x)x^{m-2} + 2m(1-x)x^{m-1} + 2x^m = 0$$

$$\Rightarrow (m^2 - 3m + 2)x^m - 2(m^2 - 2m)x^{m-1} = 0$$

$$\Rightarrow m = 2$$

故可得一補解為 $y_1 = x^2$

(2) 令另一補解 $y_2 = vy_1 \Rightarrow y_2' = v'y_1 + vy_1'$

$$\Rightarrow y_2'' = v''y_1 + 2v'y_1' + vy_1''$$

代入 ODE: $(x^2 - 2x)y'' + 2(1-x)y' + 2y = 0$

$$\text{可得 } (x^2 - 2x)(v''y_1 + 2v'y_1' + vy_1'') + 2(1-x)(v'y_1 + vy_1') + 2vy_1 = 0$$

$$\Rightarrow (x^2 - 2x)(v''y_1 + 2v'y_1') + 2(1-x)(v'y_1) = 0$$

$$\Rightarrow (x^4 - 2x^3)v'' + (2x^3 - 6x^2)v' = 0$$

$$\text{令 } z = v' \Rightarrow z' + \frac{2x^3 - 6x^2}{x^4 - 2x^3}z = 0 \Rightarrow \frac{1}{z}dz = -\frac{2x - 6}{x^2 - 2x}dx$$

$$\Rightarrow \frac{1}{z} dz = \left(-\frac{3}{x} + \frac{1}{x-2} \right) dx$$

$$\Rightarrow \ln|z| = -3 \ln|x| + \ln|x-2| + \ln c_1$$

$$\Rightarrow z = c_1 \frac{x-2}{x^3}$$

$$\Rightarrow v' = c_1 \left(\frac{1}{x^2} - \frac{2}{x^3} \right)$$

$$\Rightarrow v = c_1 \left(-\frac{1}{x} + \frac{1}{x^2} \right) + c_2$$

$$y_2 = vx^2 = v = -c_1(x-1) + c_2x^2 = c_3 \cdot (x-1) + c_2x^2$$

\therefore 另一補解為 $(x-1)$

5. 紿一微分方程 $\ddot{y}(t) + \omega^2 y(t) = F \cos \gamma t$ ，其中 ω 、 γ 、 F 為常數且 $\gamma \neq \omega$

- (1) 試求此微分方程之補解 $y_h(t) = ?$ (3%)
- (2) 試求 Wronskian，即 $W(y_1, y_2) = ?$ (2%)
- (3) 試求微分方程之通解 $y(t) = ?$ (5%)
- (4) 若給初始條件 $y(0) = 0$ 與 $\dot{y}(0) = 0$ ，則 $y(t) = ?$ (3%)
- (5) 當 $\gamma = \omega$ 時，則 $\lim_{\gamma \rightarrow \omega} y(t) = ?$ (3%)

(1) $\ddot{y}(t) + \omega^2 y(t) = F \cos(\gamma t)$

令 $y = e^{\lambda t} \Rightarrow (\lambda^2 + \omega^2)e^{\lambda t} = 0 \Rightarrow \lambda = \pm \omega i$

$\therefore y_h = c_1 \cos \omega t + c_2 \sin \omega t$

(2) 令 $y_1 = \cos \omega t$, $y_2 = \sin \omega t$

$$W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} \cos \omega t & \sin \omega t \\ -\omega \sin \omega t & \omega \cos \omega t \end{vmatrix} = \omega$$

(3) 使用待定係數法來求其特解

令 $y_p = A \cos \gamma t + B \sin \gamma t$

$$y'_p = -A \gamma \sin \gamma t + B \gamma \cos \gamma t$$

$$y''_p = -A \gamma^2 \cos \gamma t - B \gamma^2 \sin \gamma t \quad \text{代回 ODE 後可得}$$

$$\begin{aligned} & -A \gamma^2 \cos \gamma t - B \gamma^2 \sin \gamma t + \omega^2 \cdot (A \cos \gamma t + B \sin \gamma t) = F \cos \gamma t \\ & \Rightarrow A(\omega^2 - \gamma^2) \cos \gamma t + B(\omega^2 - \gamma^2) \sin \gamma t = F \cos \gamma t \\ & \Rightarrow A = \frac{F}{\omega^2 - \gamma^2}, \quad B = 0 \end{aligned}$$

$$y_p = \frac{F}{\omega^2 - \gamma^2} \cos \gamma t$$

$$\therefore y = y_h + y_p = c_1 \cos \omega t + c_2 \sin \omega t + \frac{F}{\omega^2 - \gamma^2} \cos \gamma t$$

(4) 由 $y(0) = 0 \Rightarrow c_1 = -\frac{F}{\omega^2 - \gamma^2}$

$$\dot{y}(0) = 0 \Rightarrow c_2 = 0$$

$$\therefore y = \frac{F}{\omega^2 - \gamma^2} (\cos \gamma t - \cos \omega t)$$

(5) 當 $\gamma = \omega$ 時

$$\lim_{\gamma \rightarrow \omega} y(t) = \lim_{\gamma \rightarrow \omega} \frac{F}{\omega^2 - \gamma^2} (\cos \gamma t - \cos \omega t) = \lim_{\gamma \rightarrow \omega} \frac{F \cdot t}{2\gamma} \sin \gamma t = \frac{F \cdot t}{2\omega} \sin \omega t$$