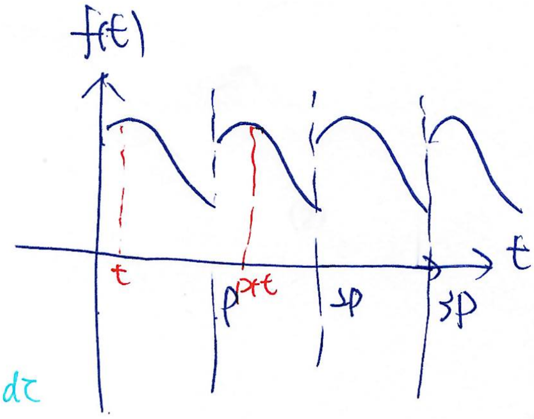
○週期函數之 Laplace transform

$$f(t) = f(p+t) = f(2p+t) = \cdots$$



週期 P

$$\therefore \mathcal{L}[f(t)] = \int_0^\infty f(t)\cdot e^{-st}\,dt$$

$$= \int_0^p f(t)\cdot e^{-st}\,dt \qquad \text{令 } t=\tau$$

$$+ \int_p^{2p} f(t)\,e^{-st}\,dt \qquad \text{令 } t=\tau+p \Rightarrow dt=d\tau$$

$$f(t)=f(\tau+p)=f(\tau)$$

$$+ \int_{2p}^{3p} f(t)\,e^{-st}\,dt \qquad \text{令 } t=\tau+2p \Rightarrow dt=d\tau$$

$$f(t)=f(\tau+2p)=f(\tau)$$

$$+ \cdots$$

$$= \int_0^p f(\tau)\,e^{-s\tau}\,d\tau$$

$$+ \int_0^p f(\tau)\,e^{-s(\tau+p)}\,d\tau \qquad e^{-s\tau}\cdot e^{-sp}$$

$$+ \int_0^p f(\tau)\,e^{-s(\tau+2p)}\,d\tau$$

$$+ \cdots$$

$$= \int_0^p f(\tau)e^{-s\tau}d\tau + e^{-sp}\cdot\int_0^p f(\tau)e^{-s\tau}d\tau$$

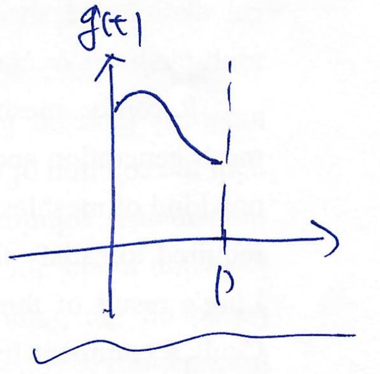
$$+ e^{-2sp}\cdot\int_0^p f(\tau)e^{-s\tau}d\tau$$

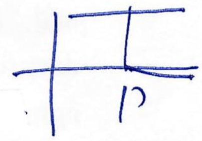
$$+ \cdots$$

$$= (1+e^{-sp}+e^{-2sp}+e^{-3sp}+\cdots)\cdot\int_0^p f(\tau)\,e^{-s\tau}d\tau$$

$$= \boxed{\frac{1}{1-e^{-sp}}}\int_0^p f(\tau)\,e^{-s\tau}d\tau \quad \#$$

$$f(t) = g(t)\cdot u(t)$$
$$+ g(t-p)\cdot u(t-p)$$
$$+ g(t-2p)\cdot u(t-2p)$$
$$+ g(t-3p)\cdot u(t-3p)$$
$$+ \text{---}$$





$$\mathcal{L}[f(t)] = \mathcal{L}[g(t)] + \mathcal{L}[g(t-p)\cdot u(t-p)]$$
$$+ \mathcal{L}[g(t-2p)\cdot u(t-2p)]$$
$$+ \mathcal{L}[g(t-3p - u(t-3p)]$$
$$+ \text{---}$$

$$= \mathcal{L}[g(t)] + e^{-sp}\cdot\mathcal{L}[g(t)] + e^{-2sp}\cdot\mathcal{L}[g(t)]$$
$$+ e^{-3sp}\cdot\mathcal{L}[g(t)]$$
$$+ \text{---}$$

$$= (1 + e^{-sp} + e^{-2sp} + e^{-3sp} + \cdots)\cdot\mathcal{L}[g(t)]$$

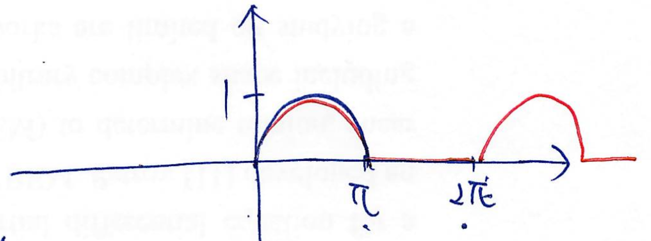
$$= \frac{1}{1-e^{-sp}}\cdot\mathcal{L}[g(t)]$$

Ex、已知在區間 $(0,\pi)$ 有 $f(t)=\sin t$

另外在區間 $(\pi, 2\pi)$ 有 $f(t)=0$

並且 $f(t)=f(t+2\pi)$，試問：

$\mathcal{L}[f(t)]=?$



週期 $P=2\pi$

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-sP}}\int_0^P f(t)\cdot e^{-st}\,dt$$

$$= \frac{1}{1-e^{-2\pi s}}\int_0^{\pi} \sin t\cdot e^{-st}\,dt$$

$$= \frac{1}{1-e^{-2\pi s}}\cdot\frac{e^{-\pi s}+1}{s^2+1}$$

○ 褶積 (convolution integral:)

$$f(t)*g(t)=\int_0^t f(t-\tau)g(\tau)\,d\tau=\int_0^t f(\tau)g(t-\tau)\,d\tau$$

令 $u=t-\tau \Rightarrow du=d\tau$

$$\therefore\ f(t)*g(t)=\int_t^0 f(u)g(t-u)(-du)$$

$$=\int_0^t f(u)g(t-u)\,du$$

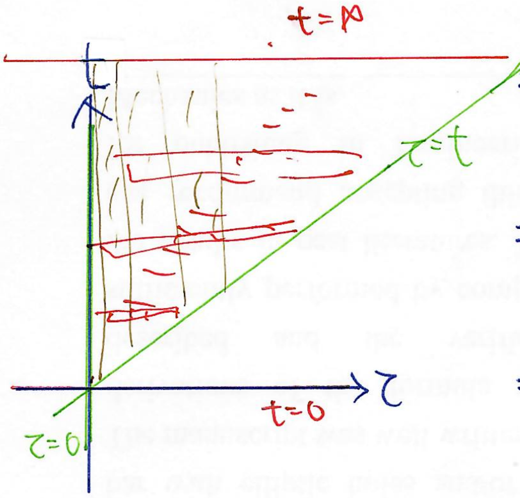
$$=g(t)*f(t)$$

○ 若 $\mathcal{L}[f(t)]=F(s)$, $\mathcal{L}[g(t)]=G(s)$

則 $\mathcal{L}[f(t)*g(t)]=F(s)\cdot G(s)$

proof:

$$\mathcal{L}[f(t)*g(t)]=\int_0^\infty\left[\int_0^t f(\tau)g(t-\tau)\,d\tau\right]\cdot e^{-st}\,dt$$



$$=\int_{t=0}^{\infty}\int_{\tau=0}^{\tau=t} f(\tau)g(t-\tau)\cdot e^{-st}\,d\tau\,dt$$

$$=\int_{\tau=0}^{\tau=\infty}\int_{t=\tau}^{t=\infty} f(\tau)g(t-\tau)\,e^{-st}\,dt\,d\tau \quad (令\ u=t-\tau)$$

$$=\int_0^\infty\int_0^\infty f(\tau)g(u)\,e^{-s(u+\tau)}\,du\,d\tau$$

$$=\int_0^\infty\int_0^\infty f(\tau)g(u)\cdot e^{-su}\cdot e^{-s\tau}\,du\,d\tau$$

$$=\int_0^\infty f(\tau)e^{-s\tau}\cdot\left[\int_0^\infty g(u)e^{-su}\,du\right]d\tau$$

$$=\int_0^\infty f(\tau)e^{-s\tau}\,d\tau\cdot\int_0^\infty g(u)e^{-su}\,du$$

$$=F(s)\cdot G(s)$$

Ex.

$$\mathcal{L}\left[\int_0^t \sin 3(\alpha - t)\cos 2\alpha \, d\alpha\right] = ?$$

$f(t) * g(t)$

$f(t-\alpha) \quad g(\alpha)$

$$\int_0^t \sin 3(\alpha - t)\cos 2\alpha \, d\alpha = -\int_0^t \sin 3(t-\alpha)\cos 2\alpha \, d\alpha$$

$$= -\sin 3t * \cos 2t$$

$$= -f(t) * g(t)$$

$\therefore f(t) = \sin 3t$

$g(t) = \cos 2\alpha$

$\therefore f(t) = \sin 3t$

$g(t) = \cos 2t$

又 $\mathcal{L}[f(t)] = F(s) = \dfrac{3}{s^2+9}$

$\mathcal{L}[g(t)] = G(s) = \dfrac{s}{s^2+4}$

$\therefore \mathcal{L}\left[\int_0^t \sin 3(\alpha - t)\cos 2\alpha \cdot d\alpha\right]$

$$= \mathcal{L}[-f(t) * g(t)]$$

$$= -F(s) \cdot G(s)$$

$$= -\frac{3}{s^2+9} \cdot \frac{s}{s^2+4}$$

$$= -\frac{3s}{(s^2+9)(s^2+4)}$$