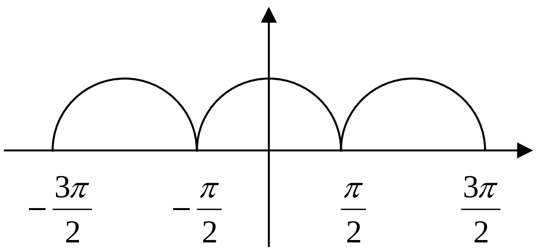
國立臺灣海洋大學河海工程學系 2022 工程數學(二) 第十次作業參考解答

系級：______________ 學號：____________ 姓名：___________

1. 試求函數 $f(x)=|\cos x|$ 複數形式之傅立葉級數。
2. 已知函數 $f(x)=f(x+4)$，且在$[0,4)$有 $f(x)=2x$ 試求 $f(x)$ 之複數形式與實數形式之傅立葉級數。
3. 已知 $f(x)$ 在 $|x|\le\dfrac{\pi}{2}$ 上有 $f(x)=\cos x$，其它區域均為 $f(x)=0$，試求函數 $f(x)$ 之傅立葉轉換，並求$\displaystyle\int_0^\infty \frac{1}{1-\omega^2}\cos\frac{\pi\omega}{2}\,d\omega=?$
4. 已知 $x\in(-1,1)$ 有 $f(x)=1+\cos\pi x$ 其它為 $f(x)=0$，試求函數之傅立葉轉換 $F(\omega)$。
5. 試求函數 $f(t)=\sin(\omega_0 t+\dfrac{\pi}{7})$ 之傅立葉轉換。
6. 已知函數 $\mathcal{F}[e^{-ax^2}]=\sqrt{\dfrac{\pi}{a}}e^{-\frac{\omega^2}{4a}}$ 試問函數 $f(x)=(x+2)e^{-a(x+2)^2}$ 之傅立葉轉換。

參考解答:

1.



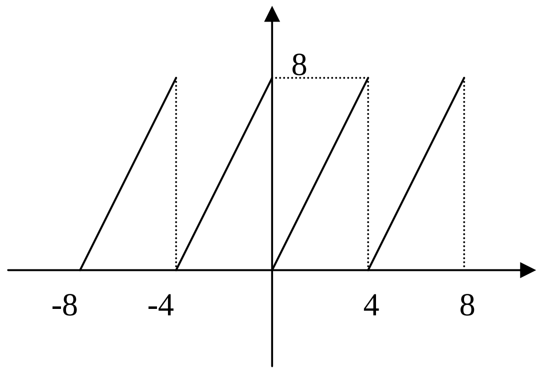
由圖可知 $f(x)$ 為偶函數且週期 $T=\pi \ \Rightarrow \omega_n=\dfrac{2n\pi}{T}=2n$

複數形式之傅立葉級數為 $f(x)=\displaystyle\sum_{n=-\infty}^{\infty}c_n e^{i\omega_n x}=\sum_{n=-\infty}^{\infty}c_n e^{2inx}$

且 $c_n=\dfrac{1}{T}\displaystyle\int_{-\infty}^{\infty} f(x)e^{-i\omega_n x}dx=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\cos x\cdot\cos 2nx\,dx=\frac{2\cdot(-1)^{n+1}}{\pi(4n^2-1)}$

$\therefore\ f(x)=\displaystyle\sum_{n=-\infty}^{\infty}c_n e^{i\omega_n x}=\frac{2}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}}{4n^2-1}e^{2inx}$

2.



由題目可知此函數週期 $T = 4 \quad \Rightarrow \omega_n = \dfrac{2n\pi}{T} = \dfrac{n\pi}{2}$

實數形式之傅立葉級數:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{T} + b_n \sin\frac{2n\pi x}{T} = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} + b_n \frac{\sin n\pi x}{2}$$

$$a_0 = \frac{1}{T}\int_0^T f(x)\,dx = \frac{1}{4}\int_0^4 2x\,dx = 4$$

$$a_n = \frac{2}{T}\int_0^T f(x)\cdot\cos\frac{2n\pi x}{T}dx = \frac{1}{2}\int_0^4 2x\cdot\cos\frac{n\pi x}{2}dx = 0$$

$$b_n = \frac{2}{T}\int_0^T f(x)\cdot\sin\frac{2n\pi x}{T}dx = \frac{1}{2}\int_0^4 2x\cdot\sin\frac{n\pi x}{2}dx = -\frac{8}{n\pi}$$

$$f(x) = 4 - \sum_{n=1}^{\infty}\frac{8}{n\pi}\sin\frac{n\pi x}{2}$$

複數形式之傅立葉級數:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n x} = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi}{2}x}$$

$$c_n = \frac{1}{T}\int_{-\infty}^{\infty} f(x)e^{-i\omega_n x}dx = \frac{1}{4}\int_0^4 2x\cdot e^{-i\frac{n\pi}{2}x}\,dx = \frac{4i}{n\pi}$$

$\because \quad n = 0$ 時 $c_n \to \infty$

$$\therefore \quad c_0 = \frac{1}{T}\int_{-\infty}^{\infty} f(x)e^{-i\omega_n x}dx = \frac{1}{4}\int_0^4 2x\,dx = 4$$

即 $\quad f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n x} = 4 + \dfrac{4i}{\pi}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\dfrac{1}{n}e^{i\frac{n\pi}{2}x}$

3. 此為偶函數

$$F(\omega) = \mathcal{F}[f(x)] = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \cos\omega x\,dx$$

$$= 2\int_0^{\frac{\pi}{2}} \cos x \cos\omega x\,dx$$

$$= \int_0^{\frac{\pi}{2}} [\cos(1+\omega)x + \cos(1-\omega)x]\,dx$$

$$= \frac{\sin\dfrac{(1+\omega)\pi}{2}}{1+\omega} + \frac{\sin\dfrac{(1-\omega)\pi}{2}}{1-\omega}$$

$$= \frac{2}{1-\omega^2}\cos\frac{\omega\pi}{2}$$

$$\therefore\ f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2}{1-\omega^2}\cos\frac{\omega\pi}{2} e^{i\omega x}\,d\omega$$

令 $x = 0 \quad \Rightarrow f(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2}{1-\omega^2}\cos\frac{\omega\pi}{2}\,d\omega$

$$\Rightarrow \int_0^{\infty} \frac{2}{1-\omega^2}\cos\frac{\omega\pi}{2}\,d\omega = \frac{\pi}{2} f(0) = \frac{\pi}{2}$$

4. $F(\omega) = \mathcal{F}[f(x)] = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx = \int_{-1}^{1} [1+\cos\pi x]e^{-i\omega x}\,dx$

$$= 2\int_0^1 [1+\cos\pi x]\cos\omega x\,dx$$

$$= \frac{2\sin\omega}{\omega} + \int_0^1 [\cos(\pi+\omega)x + \cos(\pi-\omega)x]\,dx$$

$$= \frac{2\sin\omega}{\omega} + \frac{\sin(\pi+\omega)}{\pi+\omega} + \frac{\sin(\pi-\omega)}{\pi-\omega}$$

$$= (\frac{1}{\omega} - \frac{1}{\pi+\omega} + \frac{1}{\pi-\omega})\sin\omega$$

5. $f(t) = \sin(\omega_0 t + \frac{\pi}{7}) = \sin(\omega_0 t)\cos(\frac{\pi}{7}) + \cos(\omega_0 t)\sin(\frac{\pi}{7})$

$F(\omega) = \mathcal{F}[f(t)]$

$$= \cos(\frac{\pi}{7})\cdot\frac{\pi}{i}[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)] + \sin(\frac{\pi}{7})\cdot\pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$$

6. $\mathcal{F}[f(x)] = F(\omega) \quad \Rightarrow \mathcal{F}[xf(x)] = i\frac{d}{d\omega}[F(\omega)]$

$\therefore\ \mathcal{F}[e^{-ax^2}] = \sqrt{\frac{\pi}{a}}\,e^{-\frac{\omega^2}{4a}} \quad \Rightarrow \mathcal{F}[xe^{-ax^2}] = -i\sqrt{\frac{\pi}{a}}\cdot\frac{\omega}{2a}e^{-\frac{\omega^2}{4a}}$

又 $\mathcal{F}[f(x-p)] = e^{-i\omega p}F(\omega)$

$\therefore\ \mathcal{F}[(x+2)e^{-a(x+2)^2}] = e^{i2\omega}F(\omega) = -i\sqrt{\frac{\pi}{a}}\cdot\frac{\omega}{2a}\cdot e^{i2\omega}\,e^{-\frac{\omega^2}{4a}}$