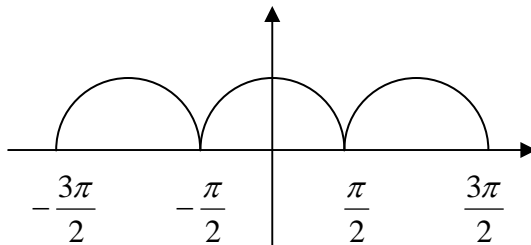


系級：_____ 學號：_____ 姓名：_____

1. 試求函數 $f(x) = |\cos x|$ 複數形式之傅立葉級數。
2. 已知函數 $f(x) = f(x+4)$ ，且在 $[0, 4)$ 有 $f(x) = 2x$ 試求 $f(x)$ 之複數形式與實數形式之傅立葉級數。
3. 已知 $f(x)$ 在 $|x| \leq \frac{\pi}{2}$ 上有 $f(x) = \cos x$ ，其它區域均為 $f(x) = 0$ ，試求函數 $f(x)$ 之傅立葉轉換，並求 $\int_0^{\infty} \frac{1}{1-\omega^2} \cos \frac{\pi\omega}{2} d\omega = ?$
4. 已知 $x \in (-1, 1)$ 有 $f(x) = 1 + \cos \pi x$ 其它為 $f(x) = 0$ ，試求函數之傅立葉轉換 $F(\omega)$ 。
5. 試求函數 $f(t) = \sin(\omega_0 t + \frac{\pi}{7})$ 之傅立葉轉換。
6. 已知函數 $\mathcal{F}[e^{-ax^2}] = \sqrt{\frac{\pi}{a}} e^{-\frac{\omega^2}{4a}}$ 試問函數 $f(x) = (x+2)e^{-a(x+2)^2}$ 之傅立葉轉換。

參考解答：

1.



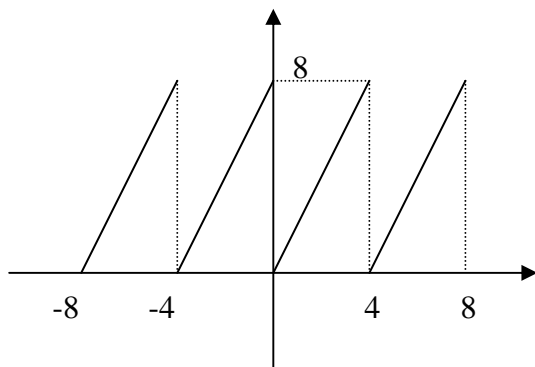
由圖可知 $f(x)$ 為偶函數且週期 $T = \pi \Rightarrow \omega_n = \frac{2n\pi}{T} = 2n$

複數形式之傅立葉級數為 $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n x} = \sum_{n=-\infty}^{\infty} c_n e^{2inx}$

$$\text{且 } c_n = \frac{1}{T} \int_{-\infty}^{\infty} f(x) e^{-i\omega_n x} dx = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos x \cdot \cos 2nx dx = \frac{2 \cdot (-1)^{n+1}}{\pi(4n^2 - 1)}$$

$$\therefore f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n x} = \frac{2}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^{n+1}}{4n^2 - 1} e^{2inx}$$

2.



由題目可知此函數週期 $T = 4 \Rightarrow \omega_n = \frac{2n\pi}{T} = \frac{n\pi}{2}$

實數形式之傅立葉級數:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{2n\pi x}{T} + b_n \sin \frac{2n\pi x}{T} = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2} + b_n \frac{\sin n\pi x}{2}$$

$$a_0 = \frac{1}{T} \int_0^T f(x) dx = \frac{1}{4} \int_0^4 2x dx = 4$$

$$a_n = \frac{2}{T} \int_0^T f(x) \cdot \cos \frac{2n\pi x}{T} dx = \frac{1}{2} \int_0^4 2x \cdot \cos \frac{n\pi x}{2} dx = 0$$

$$b_n = \frac{2}{T} \int_0^T f(x) \cdot \sin \frac{2n\pi x}{T} dx = \frac{1}{2} \int_0^4 2x \cdot \sin \frac{n\pi x}{2} dx = -\frac{8}{n\pi}$$

$$f(x) = 4 - \sum_{n=1}^{\infty} \frac{8}{n\pi} \sin \frac{n\pi x}{2}$$

複數形式之傅立葉級數:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n x} = \sum_{n=-\infty}^{\infty} c_n e^{i \frac{n\pi}{2} x}$$

$$c_n = \frac{1}{T} \int_{-\infty}^{\infty} f(x) e^{-i\omega_n x} dx = \frac{1}{4} \int_0^4 2x \cdot e^{-i \frac{n\pi}{2} x} dx = \frac{4i}{n\pi}$$

$\therefore n = 0$ 時 $c_n \rightarrow \infty$

$$\therefore c_0 = \frac{1}{T} \int_{-\infty}^{\infty} f(x) e^{-i\omega_n x} dx = \frac{1}{4} \int_0^4 2x dx = 4$$

$$\text{即 } f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n x} = 4 + \frac{4i}{\pi} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{n} e^{i \frac{n\pi}{2} x}$$

3. 此為偶函數

$$\begin{aligned} F(\omega) = \mathcal{F}[f(x)] &= \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \cos \omega x dx \\ &= 2 \int_0^{\frac{\pi}{2}} \cos x \cos \omega x dx \end{aligned}$$

$$\begin{aligned}
&= \int_0^{\frac{\pi}{2}} [\cos(1+\omega)x + \cos(1-\omega)x] dx \\
&= \frac{\sin \frac{(1+\omega)\pi}{2}}{1+\omega} + \frac{\sin \frac{(1-\omega)\pi}{2}}{1-\omega} \\
&= \frac{2}{1-\omega^2} \cos \frac{\omega\pi}{2}
\end{aligned}$$

$$\therefore f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2}{1-\omega^2} \cos \frac{\omega\pi}{2} e^{i\omega x} d\omega$$

$$\begin{aligned}
\text{令 } x=0 \quad \Rightarrow f(0) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2}{1-\omega^2} \cos \frac{\omega\pi}{2} d\omega \\
&\Rightarrow \int_0^{\infty} \frac{2}{1-\omega^2} \cos \frac{\omega\pi}{2} d\omega = \frac{\pi}{2} f(0) = \frac{\pi}{2}
\end{aligned}$$

$$\begin{aligned}
4. \quad F(\omega) = \mathcal{F}[f(x)] &= \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx = \int_{-1}^1 [1 + \cos \pi x] e^{-i\omega x} dx \\
&= 2 \int_0^1 [1 + \cos \pi x] \cos \omega x dx \\
&= \frac{2 \sin \omega}{\omega} + \int_0^1 [\cos(\pi + \omega)x + \cos(\pi - \omega)x] dx \\
&= \frac{2 \sin \omega}{\omega} + \frac{\sin(\pi + \omega)}{\pi + \omega} + \frac{\sin(\pi - \omega)}{\pi - \omega} \\
&= \left(\frac{1}{\omega} - \frac{1}{\pi + \omega} + \frac{1}{\pi - \omega} \right) \sin \omega
\end{aligned}$$

$$5. \quad f(t) = \sin(\omega_0 t + \frac{\pi}{7}) = \sin(\omega_0 t) \cos(\frac{\pi}{7}) + \cos(\omega_0 t) \sin(\frac{\pi}{7})$$

$$\begin{aligned}
F(\omega) &= \mathcal{F}[f(t)] \\
&= \cos(\frac{\pi}{7}) \cdot \frac{\pi}{i} [\delta(\omega - \omega_0) - \delta(\omega + \omega_0)] + \sin(\frac{\pi}{7}) \cdot \pi [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]
\end{aligned}$$

$$6. \quad \mathcal{F}[f(x)] = F(\omega) \quad \Rightarrow \quad \mathcal{F}[xf(x)] = i \frac{d}{d\omega} [F(\omega)]$$

$$\therefore \mathcal{F}[e^{-ax^2}] = \sqrt{\frac{\pi}{a}} e^{-\frac{\omega^2}{4a}} \quad \Rightarrow \quad \mathcal{F}[xe^{-ax^2}] = -i \sqrt{\frac{\pi}{a}} \cdot \frac{\omega}{2a} e^{-\frac{\omega^2}{4a}}$$

$$\text{又 } \mathcal{F}[f(x-p)] = e^{-i\omega p} F(\omega)$$

$$\therefore \mathcal{F}[(x+2)e^{-a(x+2)^2}] = e^{i2\omega} F(\omega) = -i \sqrt{\frac{\pi}{a}} \cdot \frac{\omega}{2a} \cdot e^{i2\omega} e^{-\frac{\omega^2}{4a}}$$