國立臺灣海洋大學河海工程學系 2022 工程數學(二) 第九次作業參考解答

系級：____________ 學號：____________ 姓名：____________

1. 試求 $f(x)=\begin{cases}1, & 0\le x<1\\ 2, & 1<x\le 2\end{cases}$ 之傅立葉全幅展開、半幅正弦展開與半幅餘弦展開並畫出相對應之圖形。
2. 試求 $f(x)=x^2+x$ 就其在區間 $(0,2)$ 上之全幅展開、半幅正弦展開與半幅餘弦展開並畫出相對應之圖形。
3. 已知若 $x>0$ 則 $f(x)=e^{-x}$，若 $x<0$ 則 $f(x)=0$，試求 $f(x)$ 之傅立葉積分，並求 $\int_0^\infty \frac{\cos 2\omega+\omega\sin 2\omega}{1+\omega^2}d\omega$ 之值。
4. 試求如下左式 $f(x)$ 之傅立葉積分，並求如下右式之積分值。

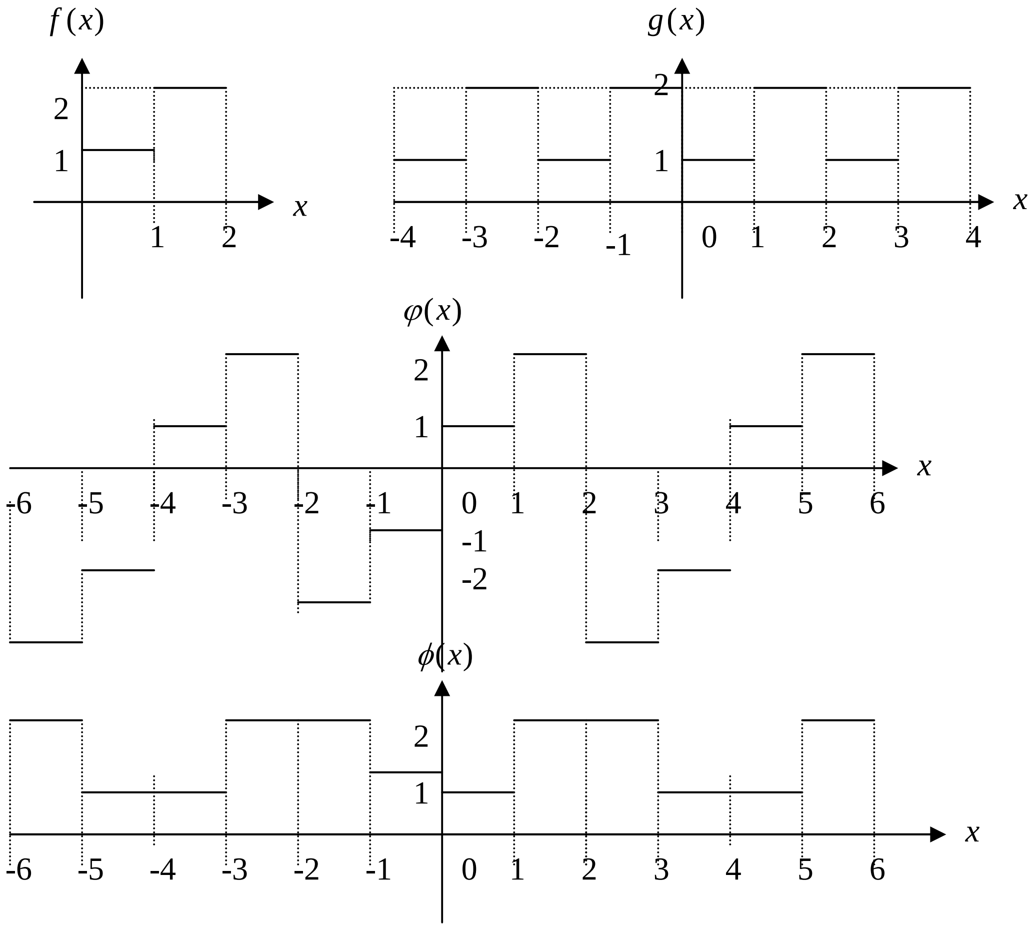
$$f(x)=\begin{cases}\cos x, & |x|<\dfrac{\pi}{2}\\ 0, & |x|>\dfrac{\pi}{2}\end{cases}\quad ;\quad \int_0^\infty \frac{1}{1-\omega^2}\cos\frac{\pi\omega}{2}d\omega$$

5. 已知函數 $f(x)=\begin{cases}1-x, & -1\le x\le 1\\ 0, & |x|>1\end{cases}$，試求函數 $f(x)$ 的傅立葉積分表示式並問

$$\int_0^\infty \frac{\sin(2x)}{x}dx=?$$

參考解答:

1.



1. 全幅展開

[0, 2] 為完整週期 $\Rightarrow T = 2$

$$g(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{T} + b_n \sin\frac{2n\pi x}{T} = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + b_n \sin(n\pi x)$$

$$a_0 = \frac{1}{T}\int_0^T f(x)\,dx = \frac{1}{2}\int_0^2 f(x)\,dx = \frac{1}{2}\left(\int_0^1 1\,dx + \int_1^2 2\,dx\right) = \frac{3}{2}$$

$$a_n = \frac{2}{T}\int_0^T f(x)\cdot\cos\frac{2n\pi x}{T}\,dx = \int_0^2 f(x)\cdot\cos(n\pi x)\,dx$$

$$= \int_0^1 \cos(n\pi x)\,dx + \int_1^2 2\cdot\cos(n\pi x)\,dx = 0$$

$$b_n = \frac{2}{T}\int_0^T f(x)\cdot\sin\frac{2n\pi x}{T}\,dx = \int_0^2 f(x)\cdot\sin(n\pi x)\,dx$$

$$= \int_0^1 \sin(n\pi x)\,dx + \int_1^2 2\sin(n\pi x)\,dx$$

$$= \frac{1}{n\pi}[(-1)^n - 1]$$

$$g(x)=\frac{3}{2}+\sum_{n=1}^{\infty}\frac{[(-1)^n-1]}{n\pi}\sin n\pi x$$

半幅正弦展開

[0, 2] 為一半週期 $\Rightarrow T=4$

$$\varphi(x)=\sum_{n=1}^{\infty}b_n\sin\frac{2n\pi x}{T}$$

$$b_n=\frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}f(x)\cdot\sin\frac{2n\pi x}{T}dx=\int_0^2 f(x)\cdot\sin\frac{n\pi x}{2}dx$$

$$=\int_0^1\sin\frac{n\pi x}{2}dx+\int_1^2 2\sin\frac{n\pi x}{2}dx$$

$$=\frac{2}{n\pi}[1-2(-1)^n+\cos(\frac{n\pi}{2})]$$

$$\varphi(x)=\sum_{n=1}^{\infty}\frac{2}{n\pi}[1-2(-1)^n+\cos(\frac{n\pi}{2})]\sin\frac{n\pi x}{2}$$

半幅餘弦展開

[0, 2] 為一半週期 $\Rightarrow T=4$

$$\phi(x)=a_0+\sum_{n=1}^{\infty}a_n\cos\frac{2n\pi x}{T}$$

$$a_0=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}f(x)\,dx=\frac{1}{2}\int_0^2 f(x)\,dx=\frac{1}{2}(\int_0^1 1\,dx+\int_1^2 2\,dx)=\frac{3}{2}$$

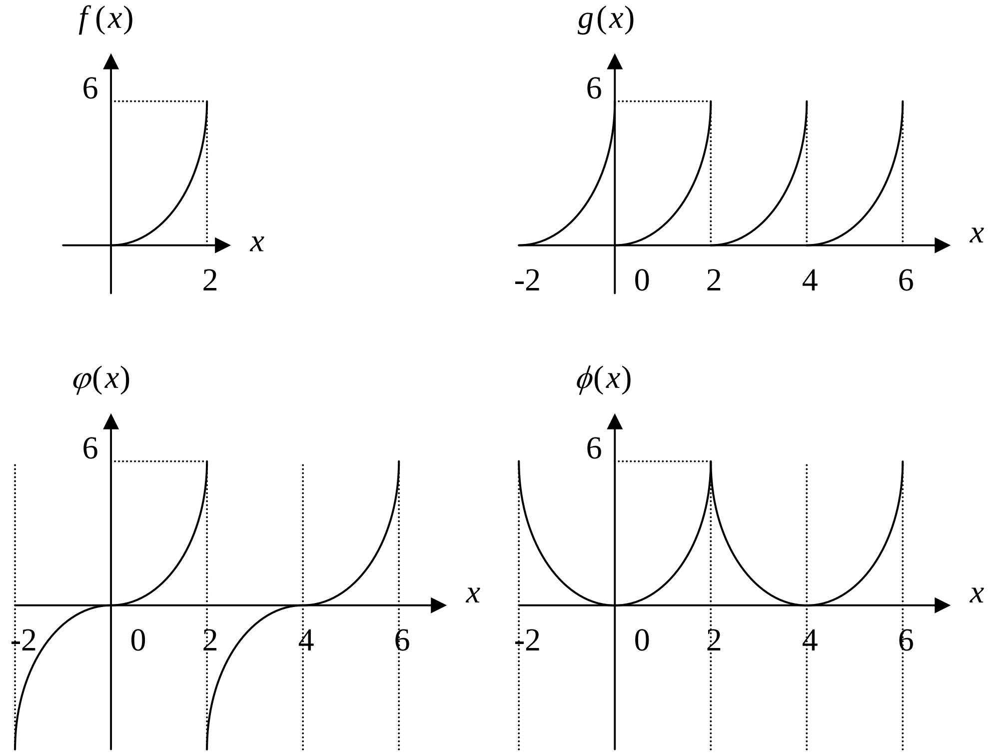
$$a_n=\frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}f(x)\cdot\cos\frac{2n\pi x}{T}dx=\int_0^2 f(x)\cdot\cos\frac{n\pi x}{2}dx$$

$$=\int_0^1\cos\frac{n\pi x}{2}dx+\int_1^2 2\cos\frac{n\pi x}{2}dx$$

$$=-\frac{2}{n\pi}\sin\frac{n\pi}{2}$$

$$\phi(x)=\frac{3}{2}-\sum_{n=1}^{\infty}\frac{2}{n\pi}\sin\frac{n\pi}{2}\cos\frac{n\pi x}{2}$$

2.



全幅展開

$(0, 2)$ 為完整週期 $\Rightarrow T = 2$

$$g(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{T} + b_n \sin\frac{2n\pi x}{T} = a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x + b_n \sin n\pi x$$

$$a_0 = \frac{1}{T}\int_0^T f(x)\,dx = \frac{1}{2}\int_0^2 (x + x^2)\,dx = \frac{7}{3}$$

$$a_n = \frac{2}{T}\int_0^T f(x)\cdot\cos\frac{2n\pi x}{T}\,dx = \int_0^2 (x + x^2)\cdot\cos n\pi x\,dx = \frac{4}{n^2\pi^2}$$

$$b_n = \frac{2}{T}\int_0^T f(x)\cdot\sin\frac{2n\pi x}{T}\,dx = \int_0^2 (x + x^2)\cdot\sin n\pi x\,dx = -\frac{6}{n\pi}$$

$$g(x) = \frac{7}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\cos n\pi x - \frac{6}{n\pi}\sin n\pi x$$

半幅正弦展開

$(0, 2)$ 為一半週期 $\Rightarrow T = 4$

$$\varphi(x) = \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi x}{T}$$

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cdot\sin\frac{2n\pi x}{T}\,dx = \int_0^2 (x + x^2)\cdot\sin n\pi x\,dx$$

$$= \frac{12}{n\pi}(-1)^{n+1} + \frac{16}{n^3\pi^3}[(-1)^n - 1]$$

$$\varphi(x) = \sum_{n=1}^{\infty} [\frac{12}{n\pi}(-1)^{n+1} + \frac{16}{n^3\pi^3}((-1)^n - 1)]\sin\frac{n\pi x}{2}$$

半幅餘弦展開

(0, 2) 為一半週期 $\Rightarrow T = 4$

$$\phi(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{T}$$

$$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\,dx = \frac{1}{2}\int_0^2 (x+x^2)\,dx = \frac{7}{3}$$

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cdot\cos\frac{2n\pi x}{T}\,dx = \int_0^2 (x+x^2)\cdot\cos\frac{n\pi x}{2}\,dx = \frac{20\cdot(-1)^n - 4}{n^2\pi^2}$$

$$\phi(x) = \frac{7}{3} + \sum_{n=1}^{\infty}\frac{20\cdot(-1)^n - 4}{n^2\pi^2}\cos\frac{n\pi x}{2}$$

3. $f(x) = \int_0^{\infty} [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\,d\omega$

且 $A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\omega x\,d\omega = \frac{1}{\pi}\int_0^{\infty} e^{-x}\cos\omega x\,d\omega = \frac{1}{\pi}\frac{1}{1+\omega^2}$

$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\omega x\,d\omega = \frac{1}{\pi}\int_0^{\infty} e^{-x}\cos\omega x\,d\omega = \frac{1}{\pi}\frac{\omega}{1+\omega^2}$

$$\therefore\ f(x) = \int_0^{\infty} [A(\omega)\cos\omega x + B(\omega)\sin\omega x]\,d\omega$$

$$= \frac{1}{\pi}\int_0^{\infty} \frac{\cos\omega x + \omega\sin\omega x}{1+\omega^2}\,d\omega$$

$$\therefore \int_0^{\infty} \frac{\cos 2\omega + \omega\sin 2\omega}{1+\omega^2}\,d\omega = \pi\cdot f(2) = \pi\cdot e^{-2}$$

4. $A(\omega) = \frac{2}{\pi}\int_0^{\infty} f(x)\cos\omega x\,dx$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos x\cdot\cos\omega x\,dx$$

$$= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} [\cos(1+\omega)x + \cos(1-\omega)x]\,dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(\frac{1+\omega}{2}\pi)}{1+\omega} + \frac{\sin(\frac{1-\omega}{2}\pi)}{1-\omega}\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos\frac{\pi\omega}{2}}{1+\omega} + \frac{\cos\frac{\pi\omega}{2}}{1-\omega}\right]$$

$$= \frac{1}{\pi}\frac{2}{1-\omega^2}\cos\frac{\pi\omega}{2}$$

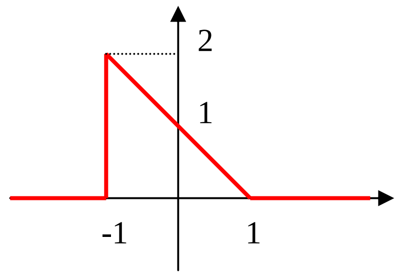
$$f(x)=\int_0^{\infty} A(\omega)\cos\omega x\, d\omega=\frac{2}{\pi}\int_0^{\infty}\frac{1}{1-\omega^2}\cos\frac{\pi\omega}{2}\cos\omega x\, d\omega$$

$$f(0)=\frac{2}{\pi}\int_0^{\infty}\frac{1}{1-\omega^2}\cos\frac{\pi\omega}{2}\, d\omega=1 \quad\Rightarrow \int_0^{\infty}\frac{1}{1-\omega^2}\cos\frac{\pi\omega}{2}\, d\omega=\frac{\pi}{2}$$

5. $f(x)=\int_0^{\infty}[A(\omega)\cos(\omega x)+B(\omega)\sin(\omega x)]d\omega$

其中 $A(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(\omega x)\,dx$

$B(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(\omega x)\,dx$



又 $f(x)=\begin{cases}1-x, & -1\le x\le 1\\ 0, & |x|>1\end{cases}$

$\therefore\ A(\omega)=\frac{1}{\pi}\int_{-1}^{1}(1-x)\cos(\omega x)\,dx=\frac{2}{\pi\omega}\sin\omega$

$B(\omega)=\frac{1}{\pi}\int_{-1}^{1}(1-x)\sin(\omega x)\,dx=\frac{2}{\pi\omega^2}(\omega\cos\omega-\sin\omega)$

$\therefore\ f(x)$ 之傅立葉積分為

$$f(x)=\int_0^{\infty}\frac{2}{\pi\omega}[\sin\omega\cdot\cos(\omega x)+\frac{1}{\omega}(\omega\cos\omega-\sin\omega)\cdot\sin(\omega x)]d\omega$$

當 $x=1$ 時，

$$\int_0^{\infty}\frac{2}{\pi\omega}[\sin\omega\cdot\cos(\omega)+\frac{1}{\omega}(\omega\cos\omega-\sin\omega)\cdot\sin(\omega)]d\omega=0$$

當 $x=-1$ 時，

$$\int_0^{\infty}\frac{2}{\pi\omega}[\sin\omega\cdot\cos(\omega)-\frac{1}{\omega}(\omega\cos\omega-\sin\omega)\cdot\sin(\omega)]d\omega=\frac{f(1^+)+f(1^-)}{2}=1$$

兩式相加可得 $\int_0^{\infty}\frac{4}{\pi\omega}\sin\omega\cdot\cos(\omega)\,d\omega=1$

$$\Rightarrow\int_0^{\infty}\frac{2\sin\omega\cdot\cos(\omega)}{\omega}d\omega=\int_0^{\infty}\frac{\sin(2\omega)}{\omega}d\omega=\int_0^{\infty}\frac{\sin(2x)}{x}dx=\frac{\pi}{2}$$