國立臺灣海洋大學河海工程學系 2022 工程數學(二) 第八次作業參考解答

系級：____________ 學號：____________ 姓名：____________

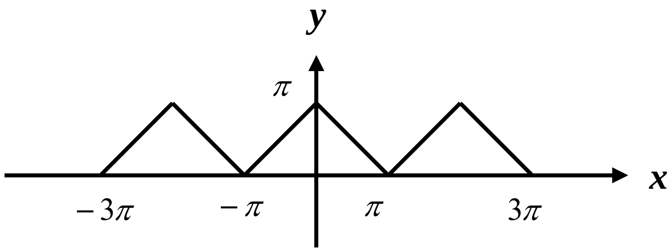
1. 已知 $x \in (-\pi, 0)$ 有 $f(x) = \pi + x$ 而 $x \in (0, \pi)$ 有 $f(x) = \pi - x$ 並且具有週期 2π
   (1) 請畫出函數 $f(x)$ 之圖形並計算 $f(x)$ 傅立葉級數展開。
   (2) 請由(1)所得之傅立葉級數，取 $-3\pi \le x \le 3\pi$ ，n 分別取 5 項、10 項、100 項繪圖。(請撰寫程式繪圖)

2. 已知 $f(x) = x$ 且 $-\pi < x < \pi$ 並且 $f(x) = f(x + 2\pi)$
   (1) 請畫出函數 $f(x)$ 之圖形並計算 $f(x)$ 傅立葉級數展開。
   (2) 請由(1)所得之傅立葉級數，取 $-3\pi \le x \le 3\pi$ ，n 分別取 10 項、20 項、100 項繪圖。(請撰寫程式繪圖)

3. 給一週期函數 $f(x) = x^2,\ -1 < x < 1$ 且 $f(x) = f(x+2)$
   (1) 試求其傅立葉級數展開。
   (2) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = ?$
   (3) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = ?$
   (4) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} = ?$
   (5) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^6} = ?$
   (6) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^8} = ?$

參考解答:

1.



可看出此為偶函數

$$\therefore\ f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{2n\pi}{T} x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx)$$

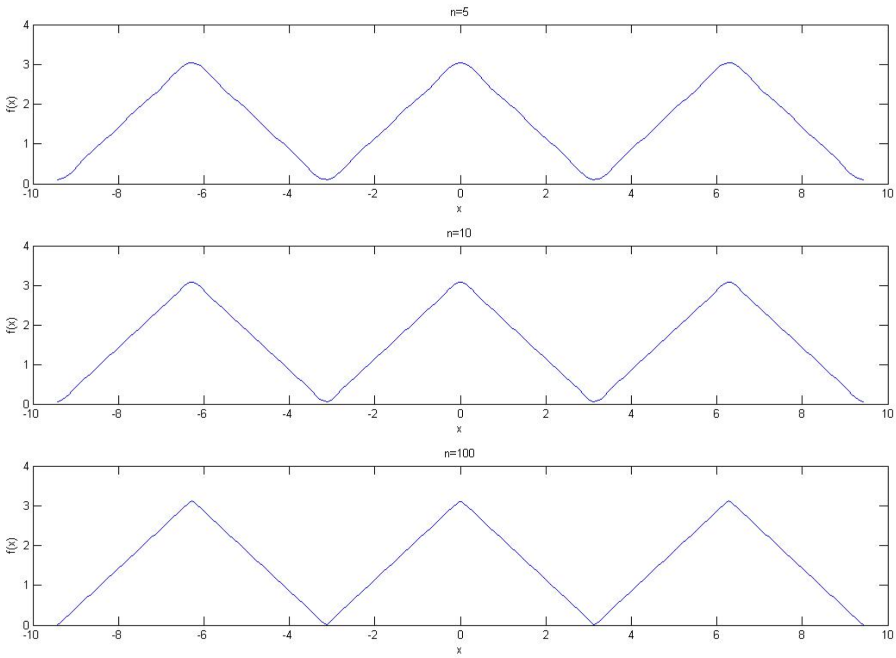
其中 $b_n = 0$

$$a_0 = \frac{2}{T}\int_0^{\frac{T}{2}} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} (\pi - x)\,dx = \frac{\pi}{2}$$

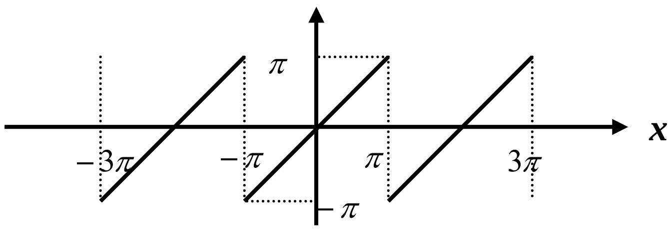
$$a_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(x)\cos(\frac{2n\pi}{T}x)\,dx = \frac{2}{\pi}\int_0^{\pi} (\pi - x)\cos(nx)\,dx$$

$$= \frac{2}{n^2\pi}[1-(-1)^n]$$

$$\therefore\ f(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]}{n^2}\cos(nx)$$



2.



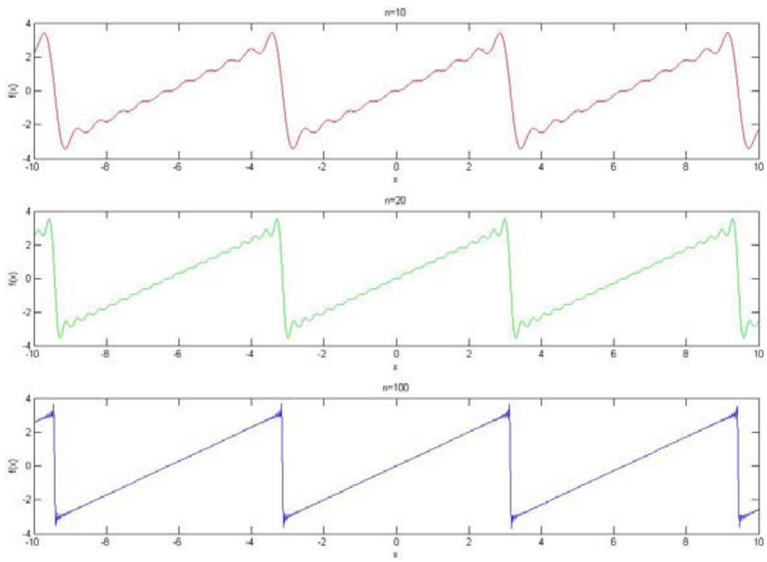
可看出此為奇函數

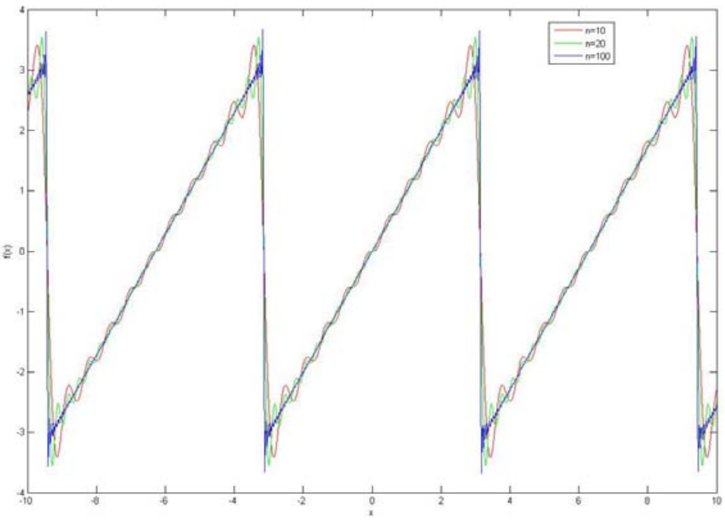
$$\therefore\ f(x)=\sum_{n=1}^{\infty} b_n \sin(\frac{2n\pi}{T}x)=\sum_{n=1}^{\infty} b_n \sin(nx)$$

其中 $a_0 = a_n = 0$

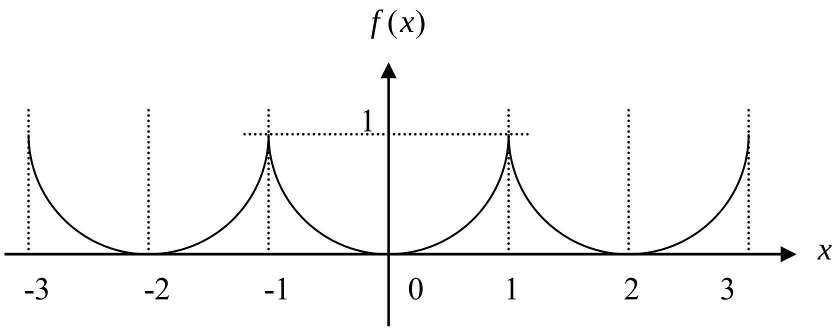
$$b_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(x)\sin(\frac{2n\pi}{T}x)\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin(nx)\,dx = \frac{2(-1)^{n+1}}{n}$$

$$\therefore\ f(x)=\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin(nx)$$





3.



(1) $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{T} + \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi x}{T}$ （已知 $T=2$，$b_n=0$）

$$\Rightarrow f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x$$

$$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\,dx = \frac{1}{2}\int_{-1}^{1} x^2\,dx = \frac{1}{3}$$

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)\cos n\pi x\,dx = \int_{-1}^{1} x^2 \cos n\pi x\,dx = \frac{4(-1)^n}{n^2\pi^2}$$

$$\therefore\ f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x$$

(2) $f(1) = 1^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi \quad \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$

(3) $f(0) = 0^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} \quad \Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$

(4) 應用 Parseval 定理: $\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f^2(x)\,dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$

可得 $\frac{1}{2}\int_{-1}^{1} x^4\,dx = \frac{1}{9} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{16}{n^4\pi^4} \quad \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$

(5) 由傅立葉級數展開可知 $f(x) = x^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x$

積分後可得

$$\frac{1}{3}x^3 = \frac{1}{3}x + \frac{4}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin n\pi x + c \quad \Rightarrow \frac{1}{3}x^3 - \frac{1}{3}x = \frac{4}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin n\pi x + c$$

當 $x = 1$ 代入，可得 $c = 0$

應用 Parseval 定理可得 $\sum_{n=1}^{\infty}\frac{16}{\pi^6 n^6} = \int_{-1}^{1}(\frac{1}{3}x^3 - \frac{1}{3}x)^2\,dx = \frac{16}{945}$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{\pi^6}{945}$$

(5) 再積分一次後可得

$$\frac{1}{12}x^4 - \frac{1}{6}x^2 = -\frac{4}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\cos n\pi x + c$$

當 $x = 1$ 代入，可得 $\frac{1}{12} - \frac{1}{6} = -\frac{4}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4} + c \quad \Rightarrow c = -\frac{7}{180}$

$\therefore \frac{1}{12}x^4 - \frac{1}{6}x^2 = -\frac{7}{180} + \frac{4}{\pi^4}\sum_{n=1}^{\infty} - \frac{(-1)^n}{n^4}\cos n\pi x$

應用 Parseval 定理可得 $\frac{1}{2}\int_{-1}^{1}(\frac{1}{12}x^4 - \frac{1}{6}x^2)^2\,dx = (-\frac{7}{180})^2 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{16}{\pi^8 n^8}$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{16}{\pi^8 n^8} = \frac{8}{4725}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^8} = \frac{\pi^8}{9450}$$