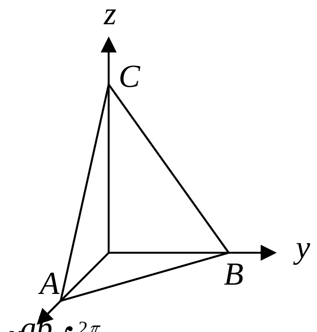
國立臺灣海洋大學河海工程學系 2022 工程數學(二) 第七次作業參考解答

系級：____________ 學號：____________ 姓名：____________

1. 試問橢圓 $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ 之面積，及其內接矩形面積之極大值。
2. 試計算線積分 $\displaystyle\oint_{x^2+y^2=1}(y^2-8y)dx+(2xy+8x)dy$ 之值。
3. 取 C 為連接 $(0,0)$、$(\frac{\pi}{2},0)$、$(\frac{\pi}{2},1)$ 之三角形封閉路徑，是請根據 C 與向量場 $\vec{F}=(y-\sin x)\vec{i}+\cos x\,\vec{j}$，驗證平面格林定理。
4. 取 C 為 $x+4y+z=12$ 在第一象限之邊界所形成之單連封閉，試計算場 $\vec{F}=(x-z)\vec{i}+(y-x)\vec{j}+(z-y)\vec{k}$ 沿路徑 C 之線積分。
5. 對於場 $\vec{F}=x^2y\vec{i}-xy^2\vec{j}+z^2\vec{k}$，試計算 $\nabla\times\vec{F}$ 在 $x^2+y^2+z^2=4$ 上半球面之面積分。
6. 試解特徵值問題: $y''+\lambda y=0$ 並滿足 $y'(0)=y(\pi)=0$ 並驗證此組函數是否具有正交性。
7. 試解特徵值問題: $y''+y'+\lambda y=0$ 並滿足 $y(0)=y(\ell)=0$
8. 試解特徵值問題: $y''+(1+\lambda)y=0$ 並滿足 $y(0)+y'(0)=y(\pi)+y'(\pi)=0$
9. 試解特徵值問題: $y^{(4)}+\lambda y''=0$ 並滿足 $y(0)=y'(0)=y(\ell)=y''(\ell)=0$

參考解答:

1\. 由格林定理可知 $\int -ydx+xdy=\iint 2\,dxdy=2A$



$$A=\frac{1}{2}\int -ydx+xdy$$

又橢圓之參數表示式為 $x=a\cos\theta \quad \Rightarrow dx=-a\sin\theta\,d\theta$

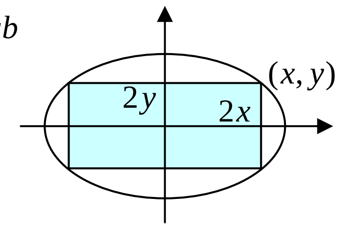
$y=b\sin\theta \quad \Rightarrow dy=b\cos\theta\,d\theta$

$$\therefore \quad A=\frac{1}{2}\int_0^{2\pi}-(b\sin\theta)(-a\sin\theta\,d\theta)+(a\cos\theta)(a\cos\theta\,d\theta)=\frac{ab}{2}\int_0^{2\pi}d\theta=\pi ab$$

在橢圓上一點 (x,y) 其內接矩形面積為 $A=4xy$

$\therefore A=4xy=4a\cos\theta\cdot b\sin\theta=2ab\sin 2\theta\le 2ab$

故內接矩形最大面積為 $2ab$



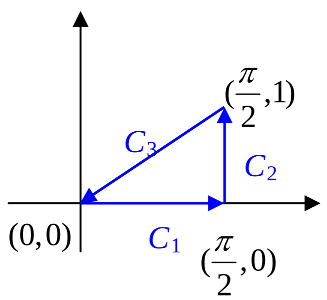
2\. 由格林定理可知

$$\oint_{x^2+y^2=1}(y^2-8y)dx+(2xy+8x)dy=\iint_{x^2+y^2\le 1}\left[\frac{\partial(2xy+8x)}{\partial x}-\frac{\partial(y^2-8y)}{\partial y}\right]dxdy$$

$$=\iint_{x^2+y^2\le 1}(2y+8-2y+8)\,dxdy$$

$$=16\pi$$

3\.



格林定理: $\int Pdx+Qdy=\iint\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=\oint \vec{F}\cdot d\vec{r}$

由面積分: $\iint\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=\iint\left(\frac{\partial(\cos x)}{\partial x}-\frac{\partial(y-\sin x)}{\partial y}\right)dxdy$

$$=-\int_0^{\frac{\pi}{2}}\int_0^{\frac{2}{\pi}x}(\sin x+1)dydx$$

$$=-\frac{2}{\pi}\int_0^{\frac{\pi}{2}}(x\sin x+x)dx$$

$$=-\left(\frac{\pi}{4}+\frac{2}{\pi}\right)$$

由線積分: $\int_C Pdx+Qdy=\int_{C_1} Pdx+Qdy+\int_{C_2} Pdx+Qdy+\int_{C_3} Pdx+Qdy$

路徑C_1:　$y=0$, $x=0\to\frac{\pi}{2}$　$\Rightarrow \int_{C_1} Pdx+Qdy=-\int_0^{\frac{\pi}{2}}\sin x\,dx=\cos x\Big|_0^{\frac{\pi}{2}}=-1$

路徑C_2:　$x=\frac{\pi}{2}$, $y=0\to 1$　$\Rightarrow \int_{C_2} Pdx+Qdy=\int_0^1\cos\frac{\pi}{2}dy=0$

路徑C_3:　$y=\frac{2}{\pi}x$　$\Rightarrow dy=\frac{2}{\pi}dx$, $x=\frac{\pi}{2}\to 0$

$$\Rightarrow \int_{C_3}(y-\sin x)\,dx+\cos x\,dy=\int_{\frac{\pi}{2}}^0(\frac{2}{\pi}x-\sin x+\frac{2}{\pi}\cos x)\,dx$$

$$=(\frac{1}{\pi}x^2+\cos x+\frac{2}{\pi}\sin x)\Big|_{\frac{\pi}{2}}^0$$

$$=1-\frac{\pi}{4}-\frac{2}{\pi}$$

$\therefore \int_C Pdx+Qdy=-\frac{\pi}{4}-\frac{2}{\pi}$

由面積分與線積分相等可驗證格林定理

4. $x+4y+z=12$ 在第一象限之邊界可得

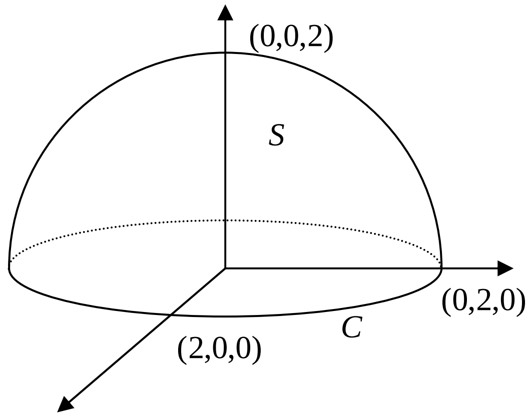
由 Stokes 定理可知 $\int_C \vec{F}\cdot d\vec{r}=\iint(\nabla\times\vec{F})\cdot\vec{n}\,dA$

ΔPQR 的單位法向量為 $\vec{n}=\frac{\overrightarrow{PQ}\times\overrightarrow{PR}}{\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|}=\frac{\vec{i}+4\vec{j}+\vec{k}}{\sqrt{18}}$

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x-z & y-x & z-y\end{vmatrix}=-(\vec{i}+\vec{j}+\vec{k})$$

$$\therefore \iint(\nabla\times\vec{F})\cdot\vec{n}\,dA=\iint\frac{-6}{\sqrt{18}}dA=\frac{-6}{\sqrt{18}}A_{\Delta PQR}=\frac{-6}{\sqrt{18}}\sqrt{18}\,A_{\Delta OPQ}=-108$$

5.



由 Stokes 定理可知 $\iint_S (\nabla\times\vec{F})\cdot\vec{n}\,dA = \int_C \vec{F}\cdot d\vec{r} = \int_C x^2y\,dx - xy^2dy$

令 $x = 2\cos\theta \quad \Rightarrow dx = -2\sin\theta\,d\theta$

$y = 2\sin\theta \quad \Rightarrow dy = 2\cos\theta\,d\theta$

$$\int_C x^2y\,dx - xy^2dy = \int_0^{2\pi} (2\cos\theta)^2(2\sin)\theta(-2\sin\theta\,d\theta) - (2\cos\theta)(2\sin\theta)^2(2\cos\theta\,d\theta)$$

$$= -32\int_0^{2\pi} \cos^2\theta\sin^2\theta\,d\theta$$

$$= -8\int_0^{2\pi} \sin^2 2\theta\,d\theta$$

$$= -8\int_0^{2\pi} \frac{1-\cos 4\theta}{2}\,d\theta$$

$$= -8\pi$$

6. $y'' + \lambda y = 0$

(1) 令 $\lambda = -k^2 \quad \Rightarrow y(x) = c_1\cosh kx + c_2\sinh kx$

$y'(0) = 0 \quad \Rightarrow c_2 = 0$

$y(\pi) = 0 \quad \Rightarrow c_1 = 0$

$\therefore\ c_1 = c_2 = 0 \quad \Rightarrow$ trivial solution

(2) 令 $\lambda = 0 \quad \Rightarrow y(x) = c_1 + c_2x$

$y'(0) = 0 \quad \Rightarrow c_2 = 0$

$y(\pi) = 0 \quad \Rightarrow c_1 = 0$

$\therefore\ c_1 = c_2 = 0 \quad \Rightarrow$ trivial solution

(3) 令 $\lambda = k^2 \quad \Rightarrow y(x) = c_1\cos kx + c_2\sin kx$

$y'(0) = 0 \quad \Rightarrow c_2 = 0$

$y(\pi) = 0 \quad \Rightarrow c_1\cos k\pi = 0$

$\Rightarrow k\pi = \frac{2n-1}{2}\pi$

$$\Rightarrow k = \frac{2n-1}{2} \quad (n = 1, 2, 3, \cdots)$$

$\therefore$ 特徵值 $\lambda = k^2 = (\frac{2n-1}{2})^2$

特徵函數 $y_n(x) = \cos(\frac{2n-1}{2}x) \quad (n = 1, 2, \cdots\cdots)$

由 $\int_0^{\pi} \cos(\frac{2n-1}{2}x)\cdot\cos(\frac{2m-1}{2}x)\,dx = \frac{1}{2}\int_0^{\pi} \cos(m+n-1)x + \cos(m-n)x\,dx = 0$

可知此組特徵函數在區間$[0,\pi]$具有正交性

7. 令 $y = e^{\mu x}$ 帶入 ODE 可得 $(\mu^2 + \mu + \lambda)e^{\mu x} = 0 \quad \Rightarrow \mu^2 + \mu + \lambda = 0$

$$\Rightarrow \mu = \frac{-1 \pm \sqrt{1-4\lambda}}{2}$$

(a) 令 $\lambda = \frac{1}{4} - k^2 \quad \Rightarrow y(x) = e^{-\frac{x}{2}}(c_1 \cosh kx + c_2 \sinh kx)$

由 $y(0) = 0 \quad \Rightarrow c_1 = 0$

$y(\ell) = 0 \quad \Rightarrow c_2 = 0$

(b) 令 $\lambda = \frac{1}{4} \quad \Rightarrow y(x) = e^{-\frac{x}{2}}(c_1 + c_2 x)$

由 $y(0) = 0 \quad \Rightarrow c_1 = 0$

$y(\ell) = 0 \quad \Rightarrow c_2 = 0$

(c) 令 $\lambda = \frac{1}{4} + k^2 \quad \Rightarrow y(x) = e^{-\frac{x}{2}}(c_1 \cos kx + c_2 \sin kx)$

由 $y(0) = 0 \quad \Rightarrow c_1 = 0$

$y(\ell) = 0 \quad \Rightarrow c_2 \sin k\ell = 0$

$\therefore$ 可知其為 $\sin k\ell = 0 \quad \Rightarrow k = \frac{n\pi}{\ell} \quad (n = 1, 2, 3, \cdots\cdots)$

故特徵值為 $\lambda_n = \frac{1}{4} + (\frac{n\pi}{\ell})^2$

特徵函數為 $y_n(x) = e^{-\frac{x}{2}} \sin \frac{n\pi x}{\ell} \quad (n = 1, 2, 3, \cdots\cdots)$

8. $y'' + (1+\lambda)y = 0$

令 $y(x) = e^{\mu x} \quad \Rightarrow \mu^2 + (1+\lambda) = 0$

$$\Rightarrow \mu = \pm\sqrt{-1-\lambda}$$

(1) 令 $\lambda = -1 - k^2 \quad \Rightarrow \mu = \pm k$

$$\Rightarrow y(x) = c_1 \cosh kx + c_2 \sinh kx$$

$y(0) + y'(0) = 0 \quad \Rightarrow c_1 + kc_2 = 0$

$y(\pi) + y'(\pi) = 0 \quad \Rightarrow c_1(\cosh k\pi + k \sinh k\pi) + c_2(\sinh k\pi + k\cosh k\pi) = 0$

可知 $\begin{vmatrix} 1 & k \\ \cosh k\pi + k\sinh k\pi & \sinh k\pi + k\cosh k\pi \end{vmatrix} = (1-k^2)\sinh k\pi = 0$

$\Rightarrow k = \pm 1$

$\therefore$ 特徵值 $\lambda = -2$

特徵函數 $y_n(x) = \cosh x \pm \sinh x$

(2) 令 $\lambda = -1 \quad \Rightarrow y(x) = c_1 + c_2 x$

$y(0) + y'(0) = 0 \quad \Rightarrow c_1 + c_2 = 0$

$y(\pi) + y'(\pi) = 0 \quad \Rightarrow (c_1 + c_2\pi) + c_2 = c_1 + c_2(1+\pi) = 0$

可知 $\begin{vmatrix} 1 & 1 \\ 1 & 1+\pi \end{vmatrix} \neq 0 \quad \Rightarrow c_1 = c_2 = 0 \quad \Rightarrow$ trivial solution

(3) 令 $\lambda = -1 + k^2 \quad \Rightarrow \mu = \pm ik$

$\Rightarrow y(x) = c_1 \cos kx + c_2 \sin kx$

$y(0) + y'(0) = 0 \quad \Rightarrow c_1 + kc_2 = 0$

$y(\pi) + y'(\pi) = 0 \quad \Rightarrow c_1(\cos k\pi - k\sin k\pi) + c_2(\sin k\pi + k\cos k\pi) = 0$

可知 $\begin{vmatrix} 1 & k \\ \cos k\pi - k\sin k\pi & \sin k\pi + k\cos k\pi \end{vmatrix} = (1+k^2)\sin k\pi = 0$

$\Rightarrow k = n \quad (n = 1, 2, \cdots\cdots)$

$\therefore$ 特徵值 $\lambda = -1 + n^2$

特徵函數 $y_n(x) = \sin(nx) - n\cos(nx) \qquad (n = 1, 2, \cdots\cdots)$

9. $y^{(4)} + \lambda y'' = 0$

(1) 令 $\lambda = -k^2 \quad \Rightarrow y(x) = c_1 + c_2 x + c_3 \cosh kx + c_4 \sinh kx$

$y(0) = 0 \quad \Rightarrow c_1 + c_3 = 0$

$y'(0) = 0 \quad \Rightarrow c_2 + kc_4 = 0$

$y(\ell) = 0 \quad \Rightarrow c_3(\cosh k\ell - 1) + c_4(\sinh k\ell - k\ell) = 0$

$y''(\ell) = 0 \quad \Rightarrow c_3 \cosh k\ell + c_4 \sinh k\ell = 0$

由 $\begin{vmatrix} \cosh k\ell - 1 & \sinh k\ell - k\ell \\ \cosh k\ell & \sinh k\ell \end{vmatrix} = k\ell\cosh k\ell - \sinh k\ell \neq 0$

$\therefore \; c_3 = c_4 = 0 \quad \Rightarrow c_1 = c_2 = 0 \quad \Rightarrow$ trivial solution

(2) 令 $\lambda = 0 \quad \Rightarrow y(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3$

$y(0) = 0 \quad \Rightarrow c_1 = 0$

$y'(0) = 0 \quad \Rightarrow c_2 = 0$

$y(\ell) = 0 \quad \Rightarrow c_3\ell^2 + c_4\ell^3 = 0$

$y''(\ell) = 0 \quad \Rightarrow 2c_3 + 6c_4\ell = 0$

由 $\begin{vmatrix} \ell^2 & \ell^3 \\ 2 & 6\ell \end{vmatrix} = 4\ell^3 \neq 0 \quad \Rightarrow c_3 = c_4 = 0 \quad \Rightarrow$ trivial solution

(3) 令 $\lambda = k^2 \quad \Rightarrow y(x) = c_1 + c_2 x + c_3 \cos kx + c_4 \sin kx$

$y(0) = 0 \quad \Rightarrow c_1 + c_3 = 0 \quad \Rightarrow c_1 = -c_3$

$y'(0) = 0 \quad \Rightarrow c_2 + kc_4 = 0 \quad \Rightarrow c_2 = -kc_4$

$y(\ell) = 0 \quad \Rightarrow c_3(\cos k\ell - 1) + c_4(\sin k\ell - k\ell) = 0$

$y''(\ell) = 0 \quad \Rightarrow c_3 \cos k\ell + c_4 \sin k\ell = 0$

由 $\begin{vmatrix} \cos k\ell - 1 & \sin k\ell - k\ell \\ \cos k\ell & \sin k\ell \end{vmatrix} = k\ell \cos k\ell - \sin k\ell = \cos k\ell (k\ell - \tan k\ell) = 0$

$\Rightarrow \tan k\ell = k\ell$

$\Rightarrow c_3 = -c_4 \tan k\ell \quad \Rightarrow c_1 = c_4 \tan k\ell$

$\Rightarrow y(x) = c_4 \tan k\ell - kc_4 x - c_4 \tan k\ell \cos kx + c_4 \sin kx$

$$= \frac{c_4}{\cos k\ell}(\sin k\ell - kx \cos k\ell - \sin k\ell \cos kx + \sin kx \cos k\ell)$$

$$= \frac{c_4}{\cos k\ell}[\sin k(x - \ell) + (k\ell - kx)\cos k\ell]$$

取特徵函數為 $y(x) = \sin k(x - \ell) + (k\ell - kx)\cos k\ell$

又 $\tan k\ell = k\ell$ 即 $\tan x = x$ 表示為 $y = \tan x$ 與 $y = x$ 的交點

這些交點為 $\{x_0, x_1, x_2, x_3, \cdots\}$ (取 $x \geq 0$)

$\therefore \quad k\ell = x_n \quad \Rightarrow k = \frac{x_n}{\ell} \quad (n = 0, 1, 2, \cdots\cdots)$

故特徵值為 $\lambda = (\frac{x_n}{\ell})^2$

特徵向量為 $y_n(x) = \sin(\frac{x_n}{\ell} x - x_n) + (x_n - \frac{x_n}{\ell} x)\cos k\ell$