國立臺灣海洋大學河海工程學系 2022 工程數學(二) 第五次作業參考解答

系級：______________　　學號：____________　　姓名：___________

1. 試求 $\int_C x^2 y\, ds$ 而 C 之路徑為 $\vec{r}(t) = a\cos t\,\vec{i} + a\sin t\,\vec{j}$ 且 $t \in [0, \pi]$

2. 試求 $\int_C (xy+z)\, ds$，其中 C 為球面 $x^2 + y^2 + z^2 = 1$ 與平面 $3z = 4y$ 之交線。

3. 試求 $\int_C y^2 dx - xy\, dy$ 而 C 為 $y = 3x - x^2$ 且 $x \in [0, 3]$

4. 試求場 $\vec{F} = xy\,\vec{i} + (3x - y^2)\,\vec{j}$，試問自 $(5, 6)$ 至 $(3, 3)$ 之直線線積分值，自 $(5, 6)$ 經 $(5, 3)$ 至 $(3, 3)$ 之折線線積分值，並問 $\vec{F}$ 是否為保守場。

5. 試計算線積分 $\int_C \vec{F} \cdot d\vec{r}$，其中 $\vec{F} = 6x^2\,\vec{i} - 2x\,\vec{j}$ 路徑 C 為由 $(5, 4) \to (1, 3) \to (0, 1) \to (5, 1)$ 之三條直線所組成。

6. 試求向量場 $\vec{F} = 3x^2 y^2\,\vec{i} + (2x^3 y - e^z)\,\vec{j} + (2z - ye^z)\,\vec{k}$，試問此向量沿任意路徑自 $(1, -2, -1)$ 至 $(-2, 3, 1)$ 之線積分值。

7. 試計算 $\int_C (5y^3 + 20x^4 y^2)\, dx + (15xy^2 + 8x^5 y - 3)\, dy$，路徑 C 為 $x^4 - 6xy^3 = 4y^2$ 由點 $(0, 0)$ 到點 $(2, 1)$

8. 試計算 $\int_C 3x^2 dx + 2yz\, dy + y^2\, dz$，其中路徑 C 為沿空間曲線

   $\vec{r}(t) = t^2\,\vec{i} + (1-2t)\,\vec{j} + (2+5t)\,\vec{k}$ 自 $(0, 1, 2)$ 到 $(1, -1, 7)$

9. 對於向量場 $\vec{F} = kxyz^2\vec{i} + (x^2 z^2 + z\cos yz)\vec{j} + (kx^2 yz + y\cos yz)\vec{k}$，試問此場為保守場之 k 值，並問自 $(1, \frac{\pi}{4}, 2)$ 至 $(2, \frac{\pi}{2}, 4)$ 之線積分。

參考解答:

1. $\vec{r}(t) = x\vec{i} + y\vec{j} = a\cos t\,\vec{i} + a\sin t\,\vec{j} \quad \Rightarrow x = a\cos t, \quad y = a\sin t$

$$ds = \sqrt{(x')^2 + (y')^2}\,dt = a\,dt$$

$$\int_C x^2 y\,ds = a^4\int_0^{\pi} \cos^2 t \cdot \sin t\,dt = -a^4\int_0^{\pi} \cos^2 t \cdot d(\cos t) = -\frac{a^4}{3}\cos^3 t\Big|_0^{\pi} = \frac{2}{3}a^4$$

2. 由 $3z = 4y \quad \Rightarrow z = \frac{4}{3}y$ 代入 $x^2 + y^2 + z^2 = 1$

可得 $x^2 + y^2 + (\frac{4}{3}y)^2 = 1 \quad \Rightarrow x^2 + \frac{25}{9}y^2 = 1$

令 $x = \cos t$，$y = \frac{3}{5}\sin t$ 可得 $z = \frac{4}{5}\sin t$

$$ds = \sqrt{(x')^2 + (y')^2 + (z')^2}\;dt = dt$$

$$\int_C (xy + z)\,ds = \int_0^{2\pi} (\cos t \cdot \frac{3}{5}\sin t + \frac{4}{5}\sin t)\,dt = \int_0^{2\pi} (\frac{3}{10}\sin 2t + \frac{4}{5}\sin t)\,dt = 0$$

3. $y = 3x - x^2 \quad \Rightarrow dy = (3 - 2x)dx$

$$\int_C y^2 dx - xy\,dy = \int_0^3 (3x - x^2)^2\,dx - x(3x - x^2)\cdot(3 - 2x)dx$$

$$= \int_0^3 (9x^2 - 6x^3 + x^4 - 9x^2 + 3x^3 + 6x^3 - 2x^4)\,dx$$

$$= \int_0^3 (3x^3 - x^4)\,dx$$

$$= \frac{3^5}{20}$$

4. 自 $(5, 6)$ 至 $(3, 3)$ 之直線 C_1 方程式為 $y = \frac{3}{2}x - \frac{3}{2} \quad \Rightarrow dy = \frac{3}{2}dx$

$$\therefore \int_{C_1} xy\,dx + (3x - y^2)\,dy = \int_5^3 x(\frac{3}{2}x - \frac{3}{2})dx + [3x - (\frac{3}{2}x - \frac{3}{2})^2]\cdot\frac{3}{2}dx = -10$$

路徑為折線之積分值為

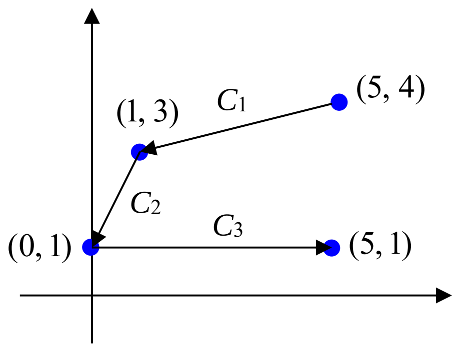
$$\int_{C_2+C_3} xy\,dx + (3x - y^2)\,dy = \int_{C_2} xy\,dx + (3x - y^2)\,dy + \int_{C_3} xy\,dx + (3x - y^2)\,dy$$

$$= \int_6^3 (15 - y^2)dy + \int_5^3 3x\,dx = -6$$

$$\because \int_{C_1} xy\,dx + (3x - y^2)\,dy \neq \int_{C_2+C_3} xy\,dx + (3x - y^2)\,dy$$

$\therefore \vec{F}$ 不是保守場

5.



$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 6x^2 & -2x & 0\end{vmatrix}=-2\vec{k} \quad\Rightarrow\ \text{此為非保守場}$$

又 $\vec{F}=6x^2\,\vec{i}-2x\,\vec{j}$ 且 $d\vec{r}=dx\vec{i}+dy\,\vec{j}\ \Rightarrow\ \int_c \vec{F}\cdot d\vec{r}=\int_c 6x^2\,dx-2x\,dy$

C_1: $(5, 4)\rightarrow(1, 3)\Rightarrow y=\frac{1}{4}(x-5)+4 \quad\Rightarrow dy=\frac{1}{4}dx$

$$\int_{C_1}\vec{F}\cdot d\vec{r}=\int_{C_1}6x^2\,dx-2x\,dy=\int_5^1(6x^2-\frac{x}{2})\,dx=-242$$

C_2: $(1, 3)\rightarrow(0, 1)\Rightarrow y=2x+1 \quad\Rightarrow dy=2dx$

$$\int_{C_2}\vec{F}\cdot d\vec{r}=\int_{C_2}6x^2\,dx-2x\,dy=\int_1^0(6x^2-4x)\,dx=0$$

C_3: $(0, 1)\rightarrow(5, 1)\Rightarrow y=1 \quad\Rightarrow dy=0dx$

$$\int_{C_3}\vec{F}\cdot d\vec{r}=\int_{C_3}6x^2\,dx-2x\,dy=\int_0^5 6x^2\,dx=250$$

$$\therefore\ \int_C\vec{F}\cdot d\vec{r}=\int_{C_1+C_2+C_3}\vec{F}\cdot d\vec{r}=8$$

6. $\because$ 沿任意路徑

$\therefore$ 先檢查是否為保守場

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 3x^2y^2 & (2x^3y-e^z) & (2z-ye^z)\end{vmatrix}=0$$

$\therefore$此為保守場並存在 $\nabla\phi=\vec{F}$

$$\frac{\partial\phi}{\partial x}=3x^2y^2 \quad\Rightarrow\phi=x^3y^2+f_1(y,z)2$$

$$\frac{\partial\phi}{\partial y}=2x^3y-e^z \quad\Rightarrow\phi=x^3y^2-ye^z+f_2(x,z)$$

$\dfrac{\partial \phi}{\partial z} = 2z - ye^z \qquad \Rightarrow \phi = z^2 - ye^z + f_3(x, y)$

比較後可得 $\phi(x, y, z) = x^3 y^2 - ye^z + z^2 + c$

自 $(1, -2, -1)$ 至 $(-2, 3, 1)$ 之線積分值為 $\phi(-2, 3, 1) - \phi(1, -2, -1) = -76 - 3e - \dfrac{2}{e}$

7 ∵ 積分路徑較為複雜

∴ 先檢查是否為保守場

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 5y^3 + 20x^4 y^2 & 15xy^2 + 8x^5 y - 3 & 0 \end{vmatrix} = 0$$

∴此為保守場並存在 $\nabla \phi = \vec{F}$

$\dfrac{\partial \phi}{\partial x} = 5y^3 + 20x^4 y^2 \qquad$ d

$\dfrac{\partial \phi}{\partial y} = 15xy^2 + 8x^5 y - 3 \qquad \Rightarrow \phi = 5xy^3 + 4x^5 y^2 - 3y + f_2(x)$

比較後可得 $\phi(x, y) = 5xy^3 + 4x^5 y^2 - 3y + c$

自 $(0, 0)$ 到點 $(2, 1)$ 之線積分值為 $\phi(2, 1) - \phi(0, 0) = 135$

8. $\vec{r}(t) = t^2\,\vec{i} + (1 - 2t)\,\vec{j} + (2 + 5t)\,\vec{k}$ 自 $(0, 1, 2)$ 到 $(1, -1, 7)$ 表示 $t: 0 \to 1$

又 $\vec{r} = x\vec{i} + y\,\vec{j} + z\,\vec{k}$ 可知

$x = t^2 \qquad \Rightarrow dx = 2t\,dt$

$y = 1 - 2t \quad \Rightarrow dy = -2\,dt$

$z = 2 + 5t \qquad \Rightarrow dz = 5\,dt$

$$\int_C 3x^2 dx + 2yz\,dy + y^2\,dz = \int_0^1 [3t^4 \cdot 2t - 4(1 - 2t)(2 + 5t) + 5(1 - 2t)^2]\,dt = 6$$

9. 此為保守場，故有 $\nabla \times \vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ kxyz^2 & x^2 z^2 + z\cos yz & kx^2 yz + y\cos yz \end{vmatrix}$$

$$= [(kx^2 z + \cos yz - yz\sin yz) - (2x^2 z + \cos yz - yz\sin yz]\vec{i}$$

$$+ (2kxyz - 2kxyz)\,\vec{j} + (2xz^2 + kxz^2)\,\vec{k} = 0$$

$$\Rightarrow k = 2$$

且 $\nabla\phi = \vec{F}$ ，即

$$\frac{\partial\phi}{\partial x} = 2xyz^2$$

$$\frac{\partial\phi}{\partial y} = x^2z^2 + z\cos yz \qquad \Rightarrow \quad \begin{aligned} \phi &= x^2yz^2 + f(y,z) \\ \phi &= x^2yz^2 + \sin yz + g(x,z) \\ \phi &= x^2yz^2 + \sin yz + h(x,y) \end{aligned}$$

$$\frac{\partial\phi}{\partial z} = 2x^2yz + y\cos yz$$

$$\therefore \quad \phi(x,y,z) = x^2yz^2 + \sin yz + c$$

場 $\vec{F}$ 自 $(1,\frac{\pi}{4},2)$ 至 $(2,\frac{\pi}{2},4)$ 之線積分為 $\phi(2,\frac{\pi}{2},4) - \phi(1,\frac{\pi}{4},2) = 31\pi - 1$