國立臺灣海洋大學河海工程學系 2022 工程數學(二) 第四次作業參考解答

系級：____________ 學號：__________ 姓名：__________

1. 已知 $f = xy^2 + 3x^2 z$，$\vec{A} = y^2\vec{i} + (y^2 - x^2)\vec{j} + 2z^2\vec{k}$，試求:

   (1) $\nabla\cdot(\nabla f)$ (2) $\nabla\times(\nabla f)$ (3) $\nabla\cdot(\nabla\times\vec{A})$ (4) $\nabla\times(\nabla\times\vec{A})$

2. 已知某山脈高度分佈為 $h(x,y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12)$，試問:
   (1) 山頂位置。
   (2) 山頂高度。
   (3) 位置 $(1,1)$ 之最陡坡度與方向
   (4) 請計算 $\nabla\cdot\nabla h$ 與 $\nabla\times\nabla h$ 之值。

3. 試求曲面 $x^3 - 2xy + z^3 + 7y + 6 = 0$ 過點 $P(1, 4, -3)$ 之切平面與法線方程式。

4. 給一向量函數 $\vec{F}(x, y, z) = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ 又

$$curl\vec{F} = \nabla\times\vec{F} = (-4y^3z^6 - 4x^5y^2)\,\vec{i} - 4z^3\,\vec{j} + (20x^4y^2z - 3x^2y^2)\,\vec{k}$$

$$div\vec{F} = \nabla\cdot\vec{F} = 2xy^3 + 8x^5yz - 6y^4z^5$$

   試找出可能之 F_1, F_2 與 F_3。(Hint: 此為非唯一解，試由觀察來得可能之解)

參考解答:

1. (1) $\nabla\cdot(\nabla f) = \nabla^2 f = \dfrac{\partial^2(xy^2+3x^2z)}{\partial x^2} + \dfrac{\partial^2(xy^2+3x^2z)}{\partial y^2} + \dfrac{\partial^2(xy^2+3x^2z)}{\partial z^2}$

$$= 2x + 6z$$

(2) $$\nabla\times(\nabla f) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial(xy^2+3x^2z)}{\partial x} & \dfrac{\partial(xy^2+3x^2z)}{\partial y} & \dfrac{\partial(xy^2+3x^2z)}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2+6xz & 2xy & 3x^2 \end{vmatrix} = 6x + 6x + 2y - 2y = 0$$

(3) $\nabla\times\vec{A}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y^2 & y^2-x^2 & 2z^2\end{vmatrix}=-2(x+y)\vec{k}$

$$\nabla\cdot(\nabla\times\vec{A})=-2\frac{\partial(x+y)}{\partial z}=0$$

(4) $\nabla\times(\nabla\times\vec{A})=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 0 & 0 & -2(x+y)\end{vmatrix}=-2\vec{i}+2\vec{j}$

2. (1) 最高點: $\frac{\partial h}{\partial x}=0 \quad \Rightarrow 10(2y-6x-18)=0$

$$\frac{\partial h}{\partial y}=0 \quad \Rightarrow 10(2x-8y+28)=0$$

解聯立可得 $x=-2$, $y=3$

(2) $h(-2,3)=10(-12-12-36+36+84+12)=720$

(3) $\nabla h=\frac{\partial h}{\partial x}\vec{i}+\frac{\partial h}{\partial y}\vec{j}=10(2y-6x-18)\,\vec{i}+10(2x-8y+28)\,\vec{j}$

在位置 $(1,1)$，$\nabla h=-220\,\vec{i}+220\,\vec{j}$

$\therefore$ 最陡方向為 $-\vec{i}+\vec{j}$，最陡坡度為 $|\nabla h|=220\sqrt{2}$

(4) $\nabla\cdot\nabla h=\frac{\partial(10(2y-6x-18))}{\partial x}+\frac{\partial(10(2x-8y+28))}{\partial y}=-140$

$$\nabla\times\nabla h=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 10(2y-6x-18) & 10(2x-8y+28) & 0\end{vmatrix}=0$$

3. 令 $\phi(x,y,z)=x^3-2xy+z^3+7y+6=0$

$$\nabla\phi=(3x^2-2y)\vec{i}+(-2x+7)\,\vec{j}+(3z^2)\vec{k}$$

在 $(1,4,-3)$ 的法向量為 $\nabla\phi=-5\vec{i}+5\,\vec{j}+27\vec{k}$

切平面方程式為 $\nabla\phi\cdot(x-1,y-4,z+3)=0$

$$\Rightarrow -5(x-1)+5(y-4)+27(z+3)=0$$

$$\Rightarrow -5x+5y+27z=-66$$

法線方程式為 $\frac{x-1}{-5}=\frac{y-4}{5}=\frac{z+3}{27}=t$

$$\Rightarrow \quad x=-5t+1,\ y=5t+4,\ z=27t-3$$

4. $\vec{F}(x, y, z) = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

由 $\nabla \cdot \vec{F} = 2xy^3 + 8x^5yz - 6y^4z^5 \;\Rightarrow \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_2}{\partial z} = 2xy^3 + 8x^5yz - 6y^4z^5$

由 $\nabla \times \vec{F} = (-4y^3z^6 - 4x^5y^2)\,\vec{i} - 4z^3\;\vec{j} + (20x^4y^2z - 3x^2y^2)\,\vec{k}$

$$\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} = -4y^3z^6 - 4x^5y^2$$

$$\Rightarrow\; \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} = -4z^3$$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 20x^4y^2z - 3x^2y^2$$

$\because\; \dfrac{\partial F_3}{\partial y} - \dfrac{\partial F_2}{\partial z} = -4y^3z^6 - 4x^5y^2$ 存在 $4y^3z^6$ 項

$\therefore$ 由 $\dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z} = 2xy^3 + 8x^5yz - 6y^4z^5$ 可知 $\dfrac{\partial F_3}{\partial z} = -6y^4z^5$

$$\Rightarrow F_3 = -y^4z^6 + f_3(x, y)$$

由 $\dfrac{\partial F_1}{\partial z} - \dfrac{\partial F_3}{\partial x} = -4z^3$ 可得 $F_1 = -z^4 + \dfrac{\partial f_3(x, y)}{\partial x}z + f_1(x, y)$

由 $\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y} = 20x^4y^2z - 3x^2y^2$ 可得

$$F_2 = 4x^5y^2z - x^3y^2 + \frac{\partial f_3(x, y)}{\partial y}z + \int \frac{\partial f_1(x, y)}{\partial y}dx$$

又 $\dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_2}{\partial z} = 2xy^3 + 8x^5yz - 6y^4z^5$

$$\frac{\partial F_1}{\partial x} = \frac{\partial^2 f_3(x, y)}{\partial x \partial x}z + \frac{\partial f_1(x, y)}{\partial x}$$

$$\frac{\partial F_2}{\partial y} = 8x^5yz - 2x^3y + \frac{\partial^2 f_3(x, y)}{\partial y \partial y}z + \int \frac{\partial^2 f_1(x, y)}{\partial y \partial y}dx$$

$$\frac{\partial F_3}{\partial z} = -6y^4z^5$$

比較後可假設 $f_3(x, y) = 0$ 與 $\displaystyle\int \frac{\partial^2 f_1(x, y)}{\partial y \partial y}dx = 2x^3y$

$$\Rightarrow f_1(x, y) = x^2y^3$$

$\therefore\; F_1 = -z^4 + x^2y^3$, $F_2 = 4x^5y^2z$, $F_3 = -y^4z^6$