

系級：_____ 學號：_____ 姓名：_____

1. 試解：

(1) $(2x+1)^2 y'' - (12x+6)y' + 16y = 2$

(2) $x(x-1)y'' + xy' - y = 0$

(3) $x(x+1)y'' + (4x+1)y' + 2y = 2x+1$

2. (1) 已知 $y = e^{2x}$ 為方程式 $(x+2)y'' - (2x+5)y' + 2y = 0$ 之解，試求此方程式之通解。

(2) 已知微分方程 $2xy'' + (1-4x)y' + (2x-1)y = e^x$ ，試求此方程式之通解。

3. 試解：

(1) $xy'' - y' - (3+x)x^2 e^x = 0$ 且 $y(0) = 0$ 、 $y(1) = 2e$

(2) $yy'' + 3(y')^2 = 0$

(3) $xyy'' + x(y')^2 + 2yy' = 0$

參考解答：

1. (1) $(2x+1)^2 y'' - (12x+6)y' + 16y = 2$

令 $u = 2x+1 \Rightarrow \frac{du}{dx} = 2$

$$y' = \frac{dy(x)}{dx} = \frac{du}{dx} \cdot \frac{dY(u)}{du} = 2Y'(u)$$

$$y'' = \frac{dy'(x)}{dx} = \frac{du}{dx} \cdot \frac{d(2Y'(u))}{du} = 4Y''(u) \text{ 代回 ODE 可得}$$

$4u^2 Y'' - 12uY' + 16Y = 2 \longrightarrow$ 此為 Euler-Cauchy ODE

令 $Y = u^m$ 代入 ODE 可得特徵方程式

$$4m(m-1) - 12m + 16 = 0 \Rightarrow m^2 - 4m + 4 = 0$$

$$\Rightarrow (m-2)^2 = 0$$

$$\Rightarrow m = 2, 2$$

$$\therefore Y_h = C_1 u^2 + C_2 u^2 \cdot \ln u$$

令 $Y_p = v_1 Y_1 + v_2 Y_2$

可知 $W = \begin{vmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{vmatrix} = \begin{vmatrix} u^2 & u^2 \cdot \ln u \\ 2u & 2u \ln u + u \end{vmatrix} = u^3$

$$v_1' = \frac{\begin{vmatrix} 0 & Y_2 \\ f(x) & Y_2' \end{vmatrix}}{W} = \frac{\begin{vmatrix} 0 & u^2 \cdot \ln u \\ 1 & 2u \ln u + u \end{vmatrix}}{u^3} = -\frac{1}{2u^3} \ln u \Rightarrow v_1 = \frac{1}{8u^2} + \frac{1}{4u^2} \ln u$$

$$v_1' = \frac{\begin{vmatrix} Y_1 & 0 \\ Y_1' & f(x) \end{vmatrix}}{W} = \frac{\begin{vmatrix} u^2 & 0 \\ 2u & 1 \end{vmatrix}}{u^3} = \frac{1}{2u^3} \Rightarrow v_2 = -\frac{1}{4u^2}$$

$$\therefore Y_p = v_1 Y_1 + v_2 Y_2 = \left(\frac{1}{8u^2} + \frac{1}{4u^2} \ln u\right) \cdot u^2 + \left(-\frac{1}{4u^2}\right) \cdot u^2 \cdot \ln u = \frac{1}{8}$$

$$\begin{aligned} \text{故可得通解為 } Y(u) &= Y_h + Y_p = C_1 u^2 + C_2 u^2 \cdot \ln u + \frac{1}{8} \\ &= C_1 (2x+1)^2 + C_2 (2x+1)^2 \cdot \ln|2x+1| + \frac{1}{8} \end{aligned}$$

(2) $x(x-1)y'' + xy' - y = 0$

令 $a_2 = x(x-1)$, $a_1 = x$, $a_0 = -1$

由 $a_2'' - a_1' + a_0 = 2 - 1 - 1 = 0$ 可知此為正合

$$\therefore x(x-1)y'' + xy' - y = \frac{d}{dx}(b_1 y' + b_0 y) = b_1 y'' + (b_1' + b_0)y' + b_0' y$$

比較係數可知 $b_1 = x(x-1)$, $b_0 = -x+1$

$$\therefore \frac{d}{dx}[x(x-1)y' - (x-1)y] = 0 \Rightarrow x(x-1)y' - (x-1)y = C_1$$

$$\Rightarrow y' - \frac{1}{x}y = \frac{C_1}{x(x-1)} \quad \text{此為一階線性 ODE}$$

可知積分因子 $\mu = e^{-\int \frac{1}{x} dx} = e^{-\ln|x|} = \frac{1}{x}$

同乘積分因子可得 $\frac{1}{x}y' - \frac{1}{x^2}y = \frac{C_1}{x^2(x-1)}$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{x}y\right) = \frac{C_1}{x^2(x-1)}$$

$$\Rightarrow \frac{1}{x}y = C_1\left(\frac{1}{x} + \ln|x-1| - \ln|x|\right) + C_2$$

$$\Rightarrow y = C_1(1 + x \ln|x-1| - x \ln|x|) + C_2 x$$

(3) $x(x+1)y'' + (4x+1)y' + 2y = 2x+1$

令 $a_2 = x(x+1)$, $a_1 = 4x+1$, $a_0 = 2$

由 $a_2'' - a_1' + a_0 = 2 - 4 + 2 = 0$ 可知此為正合

$$\therefore x(x+1)y'' + (4x+1)y' + 2y = \frac{d}{dx}(b_1y' + b_0y) = b_1y'' + (b_1' + b_0)y' + b_0'y$$

比較係數可知 $b_1 = x(x+1)$, $b_0 = 2x$

$$\therefore \frac{d}{dx}[x(x+1)y' + 2xy] = 2x+1$$

$$\Rightarrow x(x+1)y' + 2xy = x^2 + x + C_1$$

$$\Rightarrow y' + \frac{2}{x+1}y = 1 + \frac{C_1}{x(x+1)} \quad \text{此為一階線性 ODE}$$

$$\text{可知積分因子 } \mu = e^{\int \frac{2}{x+1} dx} = e^{2\ln|x+1|} = (x+1)^2$$

$$\text{同乘積分因子可得 } (x+1)^2 y' + 2(x+1)y = (x+1)^2 + C_1 \frac{x+1}{x}$$

$$\Rightarrow \frac{d}{dx}[(x+1)^2 y] = (x+1)^2 + C_1 \frac{x+1}{x}$$

$$\Rightarrow (x+1)^2 y = \frac{1}{3}(x+1)^3 + C_1(x + \ln|x|) + C_2$$

$$\Rightarrow y = \frac{1}{3}(x+1) + \frac{1}{(x+1)^2}[C_1(x + \ln|x|) + C_2]$$

$$2. (1) (x+2)y'' - (2x+5)y' + 2y = 0 \quad \Rightarrow y'' - \frac{2x+5}{x+2}y' + \frac{2}{x+2}y = 0$$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

$$\text{且滿足 } W' - \frac{2x+5}{x+2}W = 0 \quad \Rightarrow \frac{W'}{W} = \frac{2x+5}{x+2}$$

$$\Rightarrow \ln W = \int (2 + \frac{1}{x+2}) dx = 2x + \ln|x+2| + \bar{C}_1$$

$$\Rightarrow W = e^{2x + \ln|x+2| + \bar{C}_1} = C_1(x+2)e^{2x}$$

$$\therefore y_1 y_2' - y_1' y_2 = C_1(x+2)e^{2x}$$

$$\Rightarrow e^{2x} \cdot y_2' - 2e^{2x} y_2 = C_1(x+2)e^{2x}$$

$$\Rightarrow y_2' - 2y_2 = C_1(x+2) \quad \longrightarrow \quad \text{為一階線性 ODE}$$

$$\text{積分因子為 } \mu = e^{-\int 2 dx} = e^{-2x}$$

$$\text{同乘積分因子後可得 } e^{-2x} y_2' - 2e^{-2x} y_2 = C_1(x+2)e^{-2x}$$

$$\Rightarrow \frac{d}{dx}(e^{-2x} y_2) = C_1(x+2)e^{-2x}$$

$$\Rightarrow e^{-2x} y_2 = C_1(-\frac{x}{2} - \frac{5}{4})e^{-2x} + C_2$$

$$\Rightarrow e^{-2x}y_2 = C_1\left(-\frac{x}{2} - \frac{5}{4}\right)e^{-2x} + C_2$$

$$\therefore \text{另一解為 } \frac{x}{2} + \frac{5}{4}$$

$$(2) 2xy'' + (1-4x)y' + (2x-1)y = e^x$$

$$\text{先求補解滿足 } 2xy'' + (1-4x)y' + (2x-1)y = 0$$

由觀察可得一補解 e^x

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2$$

$$\text{且滿足 } W' + \frac{1-4x}{2x}W = 0 \Rightarrow \frac{W'}{W} = -\frac{1-4x}{2x}$$

$$\Rightarrow \ln W = \int \left(2 - \frac{1}{2x}\right) dx = 2x - \frac{1}{2} \ln|2x| + \bar{C}_1$$

$$\Rightarrow W = e^{2x - \frac{1}{2} \ln|2x| + \bar{C}_1} = C_1 \frac{e^{2x}}{\sqrt{2x}}$$

$$\therefore y_1y_2' - y_1'y_2 = C_1 \frac{e^{2x}}{\sqrt{2x}}$$

$$\Rightarrow e^x \cdot y_2' - e^x y_2 = C_1 \frac{e^{2x}}{\sqrt{2x}}$$

$$\Rightarrow y_2' - y_2 = C_1 \frac{e^x}{\sqrt{2x}} \longrightarrow \text{為一階線性 ODE}$$

$$\text{積分因子為 } \mu = e^{-\int dx} = e^{-x}$$

$$\text{同乘積分因子後可得 } e^{-x}y_2' - e^{-x}y_2 = \bar{C}_1 \frac{1}{\sqrt{2x}}$$

$$\Rightarrow \frac{d}{dx}(e^{-x}y_2) = \bar{C}_1 \frac{1}{\sqrt{2x}}$$

$$\Rightarrow e^{-x}y_2 = \bar{C}_1 \sqrt{2x} + C_2$$

$$\Rightarrow y_2 = C_1 e^x \sqrt{x} + C_2 e^x$$

$$\therefore \text{另一解為 } e^x \sqrt{x}$$

$$\text{由參數變異法可令 } y_p = u_1y_1 + u_2y_2$$

$$\text{可知 } W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^x \sqrt{x} \\ e^x & e^x \left(\sqrt{x} + \frac{1}{2\sqrt{x}}\right) \end{vmatrix} = \frac{1}{2\sqrt{x}} e^{2x}$$

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{W} = \frac{\begin{vmatrix} 0 & e^x \sqrt{x} \\ \frac{e^x}{2x} & e^x(\sqrt{x} + \frac{1}{2\sqrt{x}}) \end{vmatrix}}{\frac{1}{2\sqrt{x}}e^{2x}} = -1 \Rightarrow u_1 = -x$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & \frac{f(x)}{2x} \end{vmatrix}}{W} = \frac{\begin{vmatrix} e^x & 0 \\ e^x & \frac{e^x}{2x} \end{vmatrix}}{\frac{1}{2\sqrt{x}}e^{2x}} = \frac{1}{\sqrt{x}} \Rightarrow u_2 = 2\sqrt{x}$$

$$\therefore y_p = u_1 y_1 + u_2 y_2 = (-x) \cdot e^x + 2\sqrt{x} \cdot e^x \sqrt{x} = x e^x$$

$$\text{故可得通解為 } y = y_h + y_p = C_1 e^x + C_2 e^x \sqrt{x} + x e^x = e^x (C_1 + C_2 \sqrt{x} + x)$$

3. (1) $xy'' - y' - (3+x)x^2 e^x = 0$ 且 $y(0) = 0$ 、 $y(1) = 2e$

令 $u = y' \Rightarrow u' = y''$

原式 $\Rightarrow xu' - u - (3+x)x^2 e^x = 0$

$$\Rightarrow u' - \frac{1}{x}u = (3+x)xe^x \longrightarrow \text{為一階線性 ODE}$$

積分因子為 $\mu = e^{-\int \frac{1}{x} dx} = \frac{1}{x}$

同乘積分因子後可得 $\frac{1}{x}u' - \frac{1}{x^2}u = (3+x)e^x$

$$\Rightarrow \frac{d}{dx} \left(\frac{1}{x}u \right) = (3+x)e^x$$

$$\Rightarrow \frac{1}{x}u = (2+x)e^x + \bar{C}_1$$

$$\Rightarrow u = (2+x)xe^x + \bar{C}_1 x$$

$$\Rightarrow y' = (2+x)xe^x + C_1 x^2$$

$$\Rightarrow y = x^2 e^x + C_1 x^2 + C_2$$

又 $y(0) = 0 \Rightarrow C_2 = 0$

$y(1) = 2e \Rightarrow C_1 = e$

$\therefore y = x^2(e^x + e)$

(2) $yy'' + 3(y')^2 = 0$

令 $u = y' \Rightarrow y'' = \frac{dy'}{dx} = \frac{dy}{dx} \frac{dy'}{dy} = u \frac{du}{dy}$

$$\begin{aligned} \text{原式} \Rightarrow yu \frac{du}{dy} + 3u^2 = 0 &\Rightarrow u(y \frac{du}{dy} + 3u) = 0 \\ &\Rightarrow u = 0 \quad \text{or} \quad y \frac{du}{dy} + 3u \end{aligned}$$

$$\text{當 } u = 0 \Rightarrow y' = 0 \Rightarrow y = C$$

$$\begin{aligned} \text{當 } y \frac{du}{dy} + 3u = 0 &\Rightarrow -\frac{3}{y} dy = \frac{1}{u} du \\ &\Rightarrow -\int \frac{3}{y} dy = \int \frac{1}{u} du \\ &\Rightarrow -3 \ln|y| = \ln|u| + \bar{C}_1 \\ &\Rightarrow \frac{C_1}{y^3} = u \\ &\Rightarrow \frac{C_1}{y^3} = y' \\ &\Rightarrow \int y^3 dy = \int C_1 dx \\ &\Rightarrow \frac{1}{4} y^4 = C_1 x + C_2 \end{aligned}$$

\therefore 此 ODE 之解為 $\frac{1}{4} y^4 = C_1 x + C_2$ 或 $y = C$

$$(3) \quad xyy'' + x(y')^2 + 2yy' = 0$$

$$\begin{aligned} \text{令 } y = e^z &\Rightarrow y' = \frac{de^z}{dx} = \frac{dz}{dx} \frac{de^z}{dz} = e^z z' \\ &\Rightarrow y'' = \frac{dy'}{dx} = e^z (z')^2 + e^z z'' \end{aligned}$$

$$\begin{aligned} \text{原式} \Rightarrow x e^z [e^z (z')^2 + e^z z''] + x (e^z z')^2 + 2e^z e^z z' &= 0 \\ \Rightarrow x z'' + 2x (z')^2 + 2z' &= 0 \end{aligned}$$

$$\text{令 } u = z' \Rightarrow u' = z''$$

$$\text{代入上式可得 } u' + \frac{2}{x} u = -2u^2 \longrightarrow \text{此為 Bernoulli ODE}$$

$$\Rightarrow u^{-2} u' + \frac{2}{x} u^{-1} = -2$$

$$\text{令 } v = u^{-1} \Rightarrow v' = -u^{-2} u'$$

$$\text{代入上式可得 } -v' + \frac{2}{x} v = -2 \Rightarrow v' - \frac{2}{x} v = 2 \longrightarrow \text{此為一階線性 ODE}$$

$$\text{積分因子為 } \mu = e^{-\int \frac{2}{x} dx} = \frac{1}{x^2}$$

$$\begin{aligned}
&\text{同乘積分因子後可得} \quad \frac{1}{x^2}v' - \frac{2}{x^3}v = \frac{2}{x^2} \\
&\Rightarrow \frac{d}{dx}\left(\frac{1}{x^2}v\right) = \frac{2}{x^2} \\
&\Rightarrow \frac{1}{x^2}v = -\frac{2}{x} + C_1 \\
&\Rightarrow v = -2x + C_1x^2 \\
&\Rightarrow u = \frac{1}{-2x + C_1x^2} \\
&\Rightarrow z = \int \frac{1}{-2x + C_1x^2} dx = -\frac{1}{2}\ln|x| + \frac{1}{2}\ln|C_1x - 2| + C_2 \\
&\Rightarrow \ln|y| = -\frac{1}{2}\ln|x| + \frac{1}{2}\ln|C_1x - 2| + \bar{C}_2 \\
&\Rightarrow y = C_2 \frac{\sqrt{C_1x - 2}}{\sqrt{x}}
\end{aligned}$$