# 國立臺灣海洋大學河海工程學系 2021 工程數學(一) 2A 班第七次作業參考解答

系級：__________ 學號：__________ 姓名：__________

試解:

1. $x^3y''' + x^2y'' - 2xy' + 2y = x^{-2}$
2. $x^2y'' - 3xy' + 4y = 3x^2$
3. $xy'' - y' + 2x^{-1}y = 2\cos(\ln x)$
4. $x^2y'' - 2xy' + 2y = x\ln x$
5. $x^2y'' - 4xy' + 6y = -7x^4\sin x$
6. $xy'' - y' = 2x^2e^x$
7. $x^2y'' + xy' + y = \sec(\ln x)$

**參考解答:**

1. $x^3y''' + x^2y'' - 2xy' + 2y = x^{-2}$

令 $y = x^m$ 代入 ODE 可得特徵方程式

$$m(m-1)(m-2) + m(m-1) - 2m + 2 = 0 \Rightarrow m^3 - 2m^2 - m + 2 = 0$$
$$\Rightarrow (m-1)(m+1)(m-2) = 0$$
$$\Rightarrow m = 1, -1, 2$$

$\therefore\ y_h = C_1x + C_2x^{-1} + C_3x^2$

令 $y_p = u_1y_1 + u_2y_2 + u_3y_3$

可知 $W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} x & x^{-1} & x^2 \\ 1 & -x^{-2} & 2x \\ 0 & 2x^{-3} & 2 \end{vmatrix} = -6x^{-1}$

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ \frac{f(x)}{x^2} & y_2'' & y_3'' \end{vmatrix}}{W} = \frac{\begin{vmatrix} 0 & x^{-1} & x^2 \\ 0 & -x^{-2} & 2x \\ x^{-5} & 2x^{-3} & 2 \end{vmatrix}}{-6x^{-2}} = \frac{3x^{-5}}{-6x^{-1}} = -\frac{1}{2}x^{-4} \quad \Rightarrow u_1 = \frac{1}{6}x^{-3}$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & \frac{f(x)}{x^2} & y_3'' \end{vmatrix}}{W} = \frac{\begin{vmatrix} x & 0 & x^2 \\ 1 & 0 & 2x \\ 0 & x^{-5} & 2 \end{vmatrix}}{-6x^{-1}} = \frac{-x^{-3}}{-6x^{-1}} = \frac{1}{6}x^{-2} \quad \Rightarrow u_2 = -\frac{1}{6}x^{-1}$$

$$u_3' = \frac{\begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & \frac{f(x)}{x^2} \end{vmatrix}}{W} = \frac{\begin{vmatrix} x & x^{-1} & 0 \\ 1 & -x^{-2} & 0 \\ 0 & 2x^{-3} & x^{-5} \end{vmatrix}}{-6x^{-1}} = \frac{-2x^{-6}}{-6x^{-1}} = \frac{1}{3}x^{-5} \quad \Rightarrow u_3 = -\frac{1}{12}x^{-4}$$

$$\therefore \quad y_p = u_1 y_1 + u_2 y_2 + u_3 y_3 = \frac{1}{6}x^{-3} \cdot x - \frac{1}{6}x^{-1} \cdot x^{-1} - \frac{1}{12}x^{-4} \cdot x^2 = -\frac{1}{12}x^{-2}$$

故可得通解為 $y = y_h + y_p = C_1 x + C_2 x^{-1} + C_3 x^2 - \frac{1}{12}x^{-2}$

2. $x^2 y'' - 3xy' + 4y = 3x^2$

令 $y = x^m$ 代入 ODE 可得特徵方程式

$$m(m-1) - 3m + 4 = 0 \quad \Rightarrow m^2 - 4m + 4 = 0$$
$$\Rightarrow (m-2)^2 = 0$$
$$\Rightarrow m = 2, 2$$

$$\therefore \quad y_h = C_1 x^2 + C_2 x^2 \cdot \ln x$$

令 $y_p = u_1 y_1 + u_2 y_2$

可知 $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^2 \ln x \\ 2x & 2x \ln x + x \end{vmatrix} = x^3$

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ \frac{f(x)}{x^2} & y_2' \end{vmatrix}}{W} = \frac{\begin{vmatrix} 0 & x^2 \ln x \\ 3 & 2x \ln x + x \end{vmatrix}}{x^3} = -\frac{3}{x}\ln x \quad \Rightarrow u_1 = -\frac{3}{2}(\ln x)^2$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & \frac{f(x)}{x^2} \end{vmatrix}}{W} = \frac{\begin{vmatrix} x^2 & 0 \\ 2x & 3 \end{vmatrix}}{x^3} = \frac{3}{x} \quad \Rightarrow u_2 = 3 \ln x$$

$$\therefore \quad y_p = u_1 y_1 + u_2 y_2 = -\frac{3}{2}(\ln x)^2 \cdot x^2 + 3 \ln x \cdot x^2 \ln x = \frac{3}{2}x^2 (\ln x)^2$$

故可得通解為 $y = y_h + y_p = C_1 x^2 + C_2 x^2 \cdot \ln x + \frac{3}{2}x^2 (\ln x)^2$

3. $xy'' - y' + 2x^{-1}y = 2\cos(\ln x) \quad \Rightarrow x^2 y'' - xy' + 2y = 2x\cos(\ln x)$

令 $y = x^m$ 代入 ODE 可得特徵方程式

$$m(m-1) - m + 2 = 0 \quad \Rightarrow m^2 - 2m + 2 = 0$$
$$\Rightarrow m = \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 2}}{2} = 1 \pm i$$

$\therefore\quad y_h = x[C_1\cos(\ln x) + C_2\sin(\ln x)]$

令 $y_p = u_1 y_1 + u_2 y_2$

可知 $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x\cos(\ln x) & x\sin(\ln x) \\ \cos(\ln x) - \sin(\ln x) & \sin(\ln x) + \cos(\ln x) \end{vmatrix} = x$

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ \dfrac{f(x)}{x^2} & y_2' \end{vmatrix}}{W} = \frac{\begin{vmatrix} 0 & x\sin(\ln x) \\ 2x^{-1}\cos(\ln x) & \sin(\ln x) + \cos(\ln x) \end{vmatrix}}{x} = -\frac{2}{x}\cos(\ln x)\cdot\sin(\ln x)$$

$$\Rightarrow u_1 = -2\int \frac{1}{x}\cos(\ln x)\cdot\sin(\ln x)dx = -2\int \cos(\ln x)\cdot\sin(\ln x)d(\ln x) = \frac{1}{2}\cos(2\ln x)$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & \dfrac{f(x)}{x^2} \end{vmatrix}}{W} = \frac{\begin{vmatrix} x\cos(\ln x) & 0 \\ \cos(\ln x) - \sin(\ln x) & 2x^{-1}\cos(\ln x) \end{vmatrix}}{x} = \frac{2}{x}\cos(\ln x)\cdot\cos(\ln x)$$

$$\Rightarrow u_2 = 2\int \frac{1}{x}\cos(\ln x)\cdot\cos(\ln x)dx = 2\int \cos(\ln x)\cdot\cos(\ln x)d(\ln x) = \ln x + \frac{1}{2}\sin(2\ln x)$$

$$\therefore\quad y_p = u_1 y_1 + u_2 y_2 = \frac{1}{2}\cos(2\ln x)\cdot x\cos(\ln x) + (\ln x + \frac{1}{2}\sin(2\ln x))\cdot x\sin(\ln x)$$

$$= \frac{1}{2}x\cdot\cos(\ln x) + x\ln x\cdot\sin(\ln x)$$

故可得通解為 $y = y_h + y_p = x[C_1\cos(\ln x) + C_2\sin(\ln x)] + x\ln x\cdot\sin(\ln x)$

4. $x^2 y'' - 2xy' + 2y = x\ln x$

令 $y = x^m$ 代入 ODE 可得特徵方程式

$m(m-1) - 2m + 2 = 0 \quad \Rightarrow m^2 - 3m + 2 = 0$

$\Rightarrow (m-1)(m-2) = 0$

$\Rightarrow m = 1, 2$

$\therefore\quad y_h = C_1 x + C_2 x^2$

令 $y_p = u_1 y_1 + u_2 y_2$

可知 $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2$

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ \dfrac{f(x)}{x^2} & y_2' \end{vmatrix}}{W} = \frac{\begin{vmatrix} 0 & x^2 \\ x^{-1}\ln x & 2x \end{vmatrix}}{x^2} = -\frac{1}{x}\ln x$$

$$\Rightarrow u_1 = -\int \frac{1}{x}\ln x\, dx = -\int \ln x\, d(\ln x) = -\frac{1}{2}(\ln x)^2$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & \dfrac{f(x)}{x^2} \end{vmatrix}}{W} = \frac{\begin{vmatrix} x & 0 \\ 1 & x^{-1}\ln x \end{vmatrix}}{x^2} = \frac{1}{x^2}\ln x \quad \Rightarrow u_2 = \int \frac{1}{x^2}\ln x\, dx = -\frac{1}{x} - \frac{1}{x}\ln x$$

$$\therefore \quad y_p = u_1 y_1 + u_2 y_2 = -\frac{1}{2}(\ln x)^2 \cdot x - (\frac{1}{x} + \frac{1}{x}\ln x)\cdot x^2$$

$$= -\frac{1}{2}x(\ln x)^2 - x\ln x - x$$

故可得通解為 $y = y_h + y_p = C_1 x + C_2 x^2 - \frac{1}{2}x(\ln x)^2 - x\ln x$

5. $x^2 y'' - 4xy' + 6y = -7x^4 \sin x$

令 $y = x^m$ 代入 ODE 可得特徵方程式

$$m(m-1) - 4m + 6 = 0 \quad \Rightarrow m^2 - 5m + 6 = 0$$

$$\Rightarrow (m-2)(m-3) = 0$$

$$\Rightarrow m = 2, 3$$

$\therefore \quad y_h = C_1 x^2 + C_2 x^3$

令 $y_p = u_1 y_1 + u_2 y_2$

可知 $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = x^4$

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ \dfrac{f(x)}{x^2} & y_2' \end{vmatrix}}{W} = \frac{\begin{vmatrix} 0 & x^3 \\ -7x^2 \sin x & 3x^2 \end{vmatrix}}{x^4} = 7x\sin x$$

$$\Rightarrow u_1 = 7\int x\sin x\, dx = -7x\cos x + 7\sin x$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & \dfrac{f(x)}{x^2} \end{vmatrix}}{W} = \frac{\begin{vmatrix} x^2 & 0 \\ 2x & -7x^2 \sin x \end{vmatrix}}{x^4} = -7\sin x \quad \Rightarrow u_2 = -7\int \sin x\, dx = 7\cos x$$

$$\therefore \quad y_p = u_1 y_1 + u_2 y_2 = (-7x\cos x + 7\sin x)\cdot x^2 + 7\cos x \cdot x^3 = 7x^2 \sin x$$

故可得通解為 $y = y_h + y_p = C_1 x^2 + C_2 x^3 + 7x^2 \sin x$

6. $xy'' - y' = 2x^2 e^x \quad \Rightarrow x^2 y'' - xy' = 2x^3 e^x$

令 $y = x^m$ 代入 ODE 可得特徵方程式

$m(m-1) - m = 0 \quad \Rightarrow m^2 - 2m = 0 \quad \Rightarrow m(m-2) = 0 \quad \Rightarrow m = 0, 2$

$\therefore \quad y_h = C_1 + C_2 x^2$

令 $y_p = u_1 y_1 + u_2 y_2$

可知 $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} 1 & x^2 \\ 0 & 2x \end{vmatrix} = 2x$

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ \frac{f(x)}{x^2} & y_2' \end{vmatrix}}{W} = \frac{\begin{vmatrix} 0 & x^2 \\ 2xe^x & 2x \end{vmatrix}}{2x} = -x^2 e^x \quad \Rightarrow u_1 = -\int x^2 e^x \, dx = -e^x (2 - 2x + x^2)$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & \frac{f(x)}{x^2} \end{vmatrix}}{W} = \frac{\begin{vmatrix} 1 & 0 \\ 0 & 2xe^x \end{vmatrix}}{2x} = e^x \quad \Rightarrow u_2 = \int e^x \, dx = e^x$$

$\therefore \quad y_p = u_1 y_1 + u_2 y_2 = -e^x (2 - 2x + x^2) \cdot 1 + e^x \cdot x^2 = e^x (2x - 2)$

故可得通解為 $y = y_h + y_p = C_1 + C_2 x^2 + e^x (2x - 2)$

7. $x^2 y'' + xy' + y = \sec(\ln x)$

令 $y = x^m$ 代入 ODE 可得特徵方程式

$m(m-1) + m + 1 = 0 \quad \Rightarrow m^2 + 1 = 0 \quad \Rightarrow m = \pm i$

$\therefore \quad y_h = C_1 \cos(\ln x) + C_2 \sin(\ln x)$

令 $y_p = u_1 y_1 + u_2 y_2$

可知 $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos(\ln x) & \sin(\ln x) \\ -\frac{1}{x}\sin(\ln x) & \frac{1}{x}\cos(\ln x) \end{vmatrix} = x^{-1}$

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ \frac{f(x)}{x^2} & y_2' \end{vmatrix}}{W} = \frac{\begin{vmatrix} 0 & \sin(\ln x) \\ x^{-2}\sec(\ln x) & x^{-1}\cos(\ln x) \end{vmatrix}}{x^{-1}} = -\frac{1}{x}\sec(\ln x)\sin(\ln x)$$

$$\Rightarrow u_1 = -\int \frac{1}{x}\sec(\ln x)\sin(\ln x) \, dx = -\int \frac{\sin(\ln x)}{\cos(\ln x)} \, d(\ln x) = \int \frac{1}{\cos(\ln x)} \, d(\cos(\ln x))$$

$$= \ln(\cos(\ln x))$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & \dfrac{f(x)}{x^2} \end{vmatrix}}{W} = \frac{\begin{vmatrix} \cos(\ln x) & 0 \\ -\dfrac{1}{x}\sin(\ln x) & x^{-2}\sec(\ln x) \end{vmatrix}}{x^{-1}} = x^{-1} \quad \Rightarrow u_2 = \int \frac{1}{x}\, dx = \ln x$$

$$\therefore \quad y_p = u_1 y_1 + u_2 y_2 = \ln\left|\cos(\ln x)\right| \cdot \cos(\ln x) + \ln x \cdot \sin(\ln x)$$

故可得通解為

$$y = y_h + y_p = C_1 \cos(\ln x) + C_2 \sin(\ln x) + \ln\left|\cos(\ln x)\right| \cdot \cos(\ln x) + \ln x \cdot \sin(\ln x)$$