國立臺灣海洋大學河海工程學系 2021 工程數學(一) 2A 班第六次作業參考解答

系級：______________ 學號：____________ 姓名：___________

1. 試以待定係數法求解下述微分方程式:
   (1) $y''-5y'+6y=x^3$
   (2) $y''-2y'=3x$
   (3) $y''-6y'+9y=8e^{3x}$
   (4) $y''+4y=\sin 2x$
   (5) $y''+3y'+2y=e^{2x}\cos x$
   (6) $y''-2y'+2y=2e^{x}\sin x$
   (7) $y''+2y'+y=xe^{2x}$

2. 試以參數變異法求解下述微分方程式:
   (1) $y''-4y=8x^2-2x$
   (2) $y''+4y=\tan 2x$
   (3) $y''-6y'+9y=x^{-2}e^{3x}$

參考解答:

1\. (1) $y''-5y'+6y=x^3$

令 $y=e^{\lambda x}$ 代回 ODE 可得

$(\lambda^2-5\lambda+6)e^{\lambda x}=0 \;\Rightarrow \lambda^2-5\lambda+6=0 \quad \Rightarrow (\lambda-2)(\lambda-3)=0 \quad \Rightarrow \lambda=2,3$

$\therefore \; y_h=C_1e^{2x}+C_2e^{3x}$

令 $y_p=Ax^3+Bx^2+Cx+D$ 代回 ODE 可得

$$(6Ax+2B)-5(3Ax^2+2Bx+C)+6(Ax^3+Bx^2+Cx+D)=x^3$$
$$\Rightarrow 6Ax^3+(6B-15A)x^2+(6C-10B+6A)x+(6D-5C+2B)=x^3$$

比較係數可得: $A=\dfrac{1}{6},\quad B=\dfrac{5}{12},\quad C=\dfrac{19}{36},\quad D=\dfrac{65}{216}$

$$\therefore \; y_p=\frac{1}{6}x^3+\frac{5}{12}x^2+\frac{19}{36}x+\frac{65}{216}$$

$$y=y_h+y_p=C_1e^{2x}+C_2e^{3x}+\frac{1}{6}x^3+\frac{5}{12}x^2+\frac{19}{36}x+\frac{65}{216}$$

(2) $y''-2y'=3x$

令 $y=e^{\lambda x}$ 代回 ODE 可得

$(\lambda^2-2\lambda)e^{\lambda x}=0 \;\Rightarrow \lambda^2-2\lambda=0 \quad \Rightarrow \lambda(\lambda-2)=0 \quad \Rightarrow \lambda=0,2$

$\therefore \; y_h=C_1+C_2e^{2x}$

令 $y_p = x(Ax+B)$ 代回 ODE 可得

$2A-2(2Ax+B)=3x \quad \Rightarrow -4Ax+(2A-2B)=3x$

比較係數可得： $A=-\dfrac{3}{4},\quad B=-\dfrac{3}{4}$

$\therefore\ y_p = -\dfrac{3}{4}x^2-\dfrac{3}{4}x$

$y = y_h + y_p = C_1 + C_2 e^{2x} - \dfrac{3}{4}x^2 - \dfrac{3}{4}x$

(3) $y''-6y'+9y=8e^{3x}$

令 $y=e^{\lambda x}$ 代回 ODE 可得

$(\lambda^2-6\lambda+9)e^{\lambda x}=0 \ \Rightarrow \lambda^2-6\lambda+9=0 \quad \Rightarrow (\lambda-3)^2=0 \quad \Rightarrow \lambda=3,3$

$\therefore\ y_h = C_1 e^{3x} + C_2 x e^{3x}$

令 $y_p = Ax^2e^{3x} \ \Rightarrow y_p' = A(2x+3x^2)e^{3x}$

$\Rightarrow y_p'' = A(2+12x+9x^2)e^{3x}$ 代回 ODE 可得

$A(2+12x+9x^2)e^{3x}-6A(2x+3x^2)e^{3x}+9Ax^2e^{3x}=8e^{3x} \ \Rightarrow 2A=8 \ \Rightarrow A=4$

$\therefore\ y_p = 4x^2e^{3x}$

$y = y_h + y_p = C_1e^{3x} + C_2xe^{3x} + 4x^2e^{3x}$

(4) $y''+4y=\sin 2x$

令 $y=e^{\lambda x}$ 代回 ODE 可得

$(\lambda^2+4)e^{\lambda x}=0 \ \Rightarrow \lambda^2+4=0 \quad \Rightarrow \lambda=\pm 2i$

$\therefore\ y_h = C_1\cos 2x + C_2 \sin 2x$

令 $y_p = x(A\cos 2x + B\sin 2x) = x\cdot y_h \quad \Rightarrow y_p' = y_h + x\cdot y_h'$

$\Rightarrow y_p'' = 2y_h' + x\cdot y_h''$ 代回 ODE 可得

$2y_h' + x\cdot y'' + 4x\cdot y_h = \sin 2x \quad \Rightarrow 2y_h' = \sin 2x$

$\Rightarrow 2(-2A\sin 2x + 2B\cos 2x) = \sin 2x$

$\Rightarrow A=-\dfrac{1}{4},\ B=0$

$\therefore\ y_p = -\dfrac{1}{4}x\cos 2x$

$$y = y_h + y_p = C_1\cos 2x + C_2\sin 2x - \frac{1}{4}x\cos 2x$$

(5) $y'' + 3y' + 2y = e^{2x}\cos x$

令 $y = e^{\lambda x}$ 代回 ODE 可得

$(\lambda^2 + 3\lambda + 2)e^{\lambda x} = 0 \Rightarrow \lambda^2 + 3\lambda + 2 = 0 \quad \Rightarrow (\lambda+1)(\lambda+2) = 0$

$\Rightarrow \lambda = -1, -2$

$\therefore \quad y_h = C_1 e^{-x} + C_2 e^{-2x}$

令 $y_p = e^{2x}(A\cos x + B\sin x)$

$\Rightarrow y_p' = 2e^{2x}(A\cos x + B\sin x) + e^{2x}(-A\sin x + B\cos x)$

$\Rightarrow y_p'' = 3e^{2x}(A\cos x + B\sin x) + 4e^{2x}(-A\sin x + B\cos x)$ 代回 ODE 可得

$$\begin{aligned}&3e^{2x}(A\cos x + B\sin x) + 4e^{2x}(-A\sin x + B\cos x)\\&+6e^{2x}(A\cos x + B\sin x) + 3e^{2x}(-A\sin x + B\cos x) + 2e^{2x}(A\cos x + B\sin x)\\&= e^{2x}\cos x\end{aligned}$$

$\Rightarrow 11e^{2x}(A\cos x + B\sin x) + 7e^{2x}(-A\sin x + B\cos x) = e^{2x}\cos x$

$$\Rightarrow \begin{cases} 11A + 7B = 1 \\ -7A + 11B = 0 \end{cases}$$

$\Rightarrow A = \frac{11}{170},\ B = \frac{7}{170}$

$\therefore \quad y_p = e^{2x}(\frac{11}{170}\cos x + \frac{7}{170}\sin x)$

$$y = y_h + y_p = C_1 e^{-x} + C_2 e^{-2x} + e^{2x}(\frac{11}{170}\cos x + \frac{7}{170}\sin x)$$

(6) $y'' - 2y' + 2y = 2e^{x}\sin x$

令 $y = e^{\lambda x}$ 代回 ODE 可得

$(\lambda^2 - 2\lambda + 2)e^{\lambda x} = 0 \Rightarrow \lambda^2 - 2\lambda + 2 = 0 \quad \Rightarrow \lambda = \frac{2 \pm \sqrt{(-2)^2 - 4\cdot 1\cdot 2}}{2} = 1 \pm i$

$\therefore \quad y_h = e^{x}(C_1\cos x + C_2\sin x)$

令 $y_p = x\cdot e^{x}(A\cos x + B\sin x) = x\cdot y_h$

$\Rightarrow y_p' = y_h + x\cdot y_h'$

$\Rightarrow y_p'' = y_h' + y_h' + x\cdot y_h'' = 2y_h' + x\cdot y_h''$ 代回 ODE 可得

$$(2y_h' + x\cdot y_h'') - 2(y_h + x\cdot y_h') + 2(x\cdot y_h) = 2e^x \sin x$$

$$\Rightarrow y_h' - y_h = e^x \sin x$$

$$\Rightarrow e^x(A\cos x + B\sin x) + e^x(-A\sin x + B\cos x) - e^x(A\cos x + B\sin x) = e^x \sin x$$

$$\Rightarrow A = -1,\ B = 0$$

$$\therefore\ y_p = -x\cdot e^x \cos x$$

$$y = y_h + y_p = e^x(C_1\cos x + C_2\sin x) - x\cdot e^x \cos x$$

(7) $y'' + 2y' + y = xe^{2x}$

令 $y = e^{\lambda x}$ 代回 ODE 可得

$$(\lambda^2 + 2\lambda + 1)e^{\lambda x} = 0 \quad \Rightarrow \lambda^2 + 2\lambda + 1 = 0 \quad \Rightarrow (\lambda+1)^2 = 0 \quad \Rightarrow \lambda = -1, -1$$

$$\therefore\ y_h = e^{-x}(C_1 + C_2 x)$$

令 $y_p = (Ax + B)e^{2x}$

$$\Rightarrow y_p' = (2Ax + 2B + A)e^{2x}$$

$$\Rightarrow y_p'' = (4Ax + 4B + 4A)e^{2x}$$ 代回 ODE 可得

$$(4Ax + 4B + 4A)e^{2x} + 2(2Ax + 2B + A)e^{2x} + (Ax + B)e^{2x} = xe^{2x}$$

$$\Rightarrow 9Ax + 9B + 6A = x$$

$$\Rightarrow A = \frac{1}{9},\ B = -\frac{2}{27}$$

$$\therefore\ y_p = (\frac{1}{9}x - \frac{2}{27})e^{2x}$$

$$y = y_h + y_p = e^{-x}(C_1 + C_2 x) + (\frac{1}{9}x - \frac{2}{27})e^{2x}$$

2. (1) $y'' - 4y = 8x^2 - 2x$

令 $y = e^{\lambda x}$ 代入 ODE 可得特徵方程式

$$\lambda^2 - 4 = 0 \quad \Rightarrow (\lambda + 2)(\lambda - 2) = 0 \quad \Rightarrow \lambda = -2, 2$$

$$\therefore\ y_h = C_1 e^{-2x} + C_2 e^{2x}$$

令 $y_p = u_1 y_1 + u_2 y_2$

可知 $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-2x} & e^{2x} \\ -2e^{-2x} & 2e^{2x} \end{vmatrix} = 4$

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{W} = \frac{\begin{vmatrix} 0 & e^{2x} \\ 8x^2 - 2x & 2e^{2x} \end{vmatrix}}{4} = -(2x^2 - \frac{1}{2}x)e^{2x} \quad \Rightarrow u_1 = -(x^2 - \frac{5}{4}x + \frac{5}{8})e^{2x}$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & \dfrac{f(x)}{x^2} \end{vmatrix}}{W} = \frac{\begin{vmatrix} e^{-2x} & 0 \\ -2e^{-2x} & 8x^2 - 2x \end{vmatrix}}{4} = (2x^2 - \frac{1}{2}x)e^{-2x} \Rightarrow u_2 = -(x^2 + \frac{3}{4}x + \frac{3}{8})e^{-2x}$$

$$\therefore \quad y_p = u_1 y_1 + u_2 y_2 = -(x^2 - \frac{5}{4}x + \frac{5}{8})e^{2x} \cdot e^{-2x} - (x^2 + \frac{3}{4}x + \frac{3}{8})e^{-2x} \cdot e^{2x}$$

$$= -2x^2 + \frac{1}{2}x - 1$$

故可得通解為 $y = y_h + y_p = C_1 e^{-2x} + C_2 e^{2x} - 2x^2 + \frac{1}{2}x - 1$

(2) $y'' + 4y = \tan 2x$

令 $y = e^{\lambda x}$ 代入 ODE 可得特徵方程式

$\lambda^2 + 4 = 0 \quad \Rightarrow \lambda^2 = -4 \quad \Rightarrow \lambda = \pm 2i$

$\therefore \quad y_h = C_1 \cos 2x + C_2 \sin 2x$

令 $y_p = u_1 y_1 + u_2 y_2$

可知 $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{W} = \frac{\begin{vmatrix} 0 & \sin 2x \\ \tan 2x & 2\cos 2x \end{vmatrix}}{2} = -\frac{1}{2}\tan 2x \cdot \sin 2x$$

$$\Rightarrow u_1 = \frac{1}{4}(\ln|\cos x - \sin x| - \ln|\cos x + \sin x| + \sin 2x)$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & \dfrac{f(x)}{x^2} \end{vmatrix}}{W} = \frac{\begin{vmatrix} \cos 2x & 0 \\ -2\sin 2x & \tan 2x \end{vmatrix}}{2} = \frac{1}{2}\sin 2x \quad \Rightarrow u_2 = -\frac{1}{4}\cos 2x$$

$$\therefore \quad y_p = u_1 y_1 + u_2 y_2$$

$$= \frac{1}{4}(\ln|\cos x - \sin x| - \ln|\cos x + \sin x| + \sin 2x) \cdot \cos 2x - \frac{1}{4}\cos 2x \cdot \sin 2x$$

$$= \frac{1}{4}(\ln|\cos x - \sin x| - \ln|\cos x + \sin x|) \cdot \cos 2x$$

$$= -\frac{1}{4}\cos 2x \cdot \ln\left|\frac{\cos x + \sin x}{\cos x - \sin x}\right|$$

$$= -\frac{1}{4}\cos 2x \cdot \ln\left|\frac{1 + \tan x}{1 - \tan x}\right|$$

$$= -\frac{1}{4}\cos 2x \cdot \ln\left|\tan(x + \frac{\pi}{4})\right| \qquad (\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta})$$

故可得通解為 $y = y_h + y_p = C_1\cos 2x + C_2 \sin 2x - \frac{1}{4}\cos 2x \cdot \ln\left|\tan(\frac{\pi}{4} + x)\right|$

(3) $y'' - 6y' + 9y = x^{-2}e^{3x}$

令 $y = e^{\lambda x}$ 代入 ODE 可得特徵方程式

$\lambda^2 - 6\lambda + 9 = 0 \quad \Rightarrow (\lambda - 3)^2 = 0 \quad \Rightarrow \lambda = 3, 3$

$\therefore \quad y_h = C_1 e^{3x} + C_2 x e^{3x}$

令 $y_p = u_1 y_1 + u_2 y_2$

可知 $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & (3x+1)e^{3x} \end{vmatrix} = e^{6x}$

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{W} = \frac{\begin{vmatrix} 0 & xe^{3x} \\ x^{-2}e^{3x} & (3x+1)e^{3x} \end{vmatrix}}{e^{6x}} = -x^{-1} \quad \Rightarrow u_1 = -\ln|x|$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & \frac{f(x)}{x^2} \end{vmatrix}}{W} = \frac{\begin{vmatrix} e^{3x} & 0 \\ 3e^{3x} & x^{-2}e^{3x} \end{vmatrix}}{e^{6x}} = x^{-2} \quad \Rightarrow u_2 = -\frac{1}{x}$$

$$\therefore \quad y_p = u_1 y_1 + u_2 y_2 = -\ln|x| \cdot e^{3x} - \frac{1}{x} \cdot xe^{3x} = -\ln|x| \cdot e^{3x} - e^{3x}$$

故可得通解為 $y = y_h + y_p = e^{3x}(C_1 + C_2 x - \ln|x|)$