# 國立臺灣海洋大學河海工程學系 2021 工程數學(一) 2A 班第五次作業參考解答

系級：____________　　學號：____________　　姓名：____________

1. 給一微分方程 $y''-8y'+16y=0$, 已知其解為 $y_1=e^{4x}$（重根), 試用已知一解求另一補解找出 $y_2=?$
2. 給一微分方程 $(x^2-2x)y''+2(1-x)y'+2y=0$, 已知一補解為 $y_1=x^2$, 試求另一補解 $y_2$
3. 試解下述二階以上常係數齊次 ODE
   (1) $y''-4y'+2y=0$
   (2) $y''-10y'+25y=0$
   (3) $y''+2y'+6y=0$
   (4) $y''-2y'+(\pi^2+1)y=0$； $y(\frac{1}{4})=0$， $y'(\frac{1}{4})=-\pi\sqrt[4]{4e}$
   (5) $4y''+4y'+1y=0$； $y(0)=-2$， $y(2)=e^{-1}$
   (6) $y'''+6y''+12y'+8y=0$

<參考解答>

1. $y''-8y'+16y=0$ 已知其解為 $y_1=e^{4x}$

由 Wronskian 可知

$$W(y_1,y_2)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=y_1y_2'-y_1'y_2=C_1e^{-\int a dx}$$

$$\Rightarrow y_2'-\frac{y_1'}{y_1}y_2=\frac{C_1}{y_1}e^{-\int a dx}=\frac{C_1}{y_1}e^{\int 6dx}=\frac{C_1}{y_1}e^{-6x}$$

又 $y_1=e^{4x}$ 代入後可得

$\therefore\ y_2'-4y_2=C_1e^{4x}$ ⟶ 此為一階線性 ODE

可知積分因子為 $\mu=e^{\int p(x)dx}=e^{-\int 4dx}=e^{-4x}$

$$e^{-4x}y_2'-4e^{-4x}y_2=C_1 \quad \Rightarrow \frac{d}{dx}(e^{-4x}y_2)=C_1$$

$$\Rightarrow e^{-4x}y_2=C_1x+C_2$$

$$\Rightarrow y_2=C_1xe^{4x}+C_2e^{4x}$$

$y_1=e^{4x}$ 為其一補解，故可知另一補解為 $y_2=xe^{4x}$

$x^2y''-3xy'+4y=0$

2. $(x^2-2x)y''+2(1-x)y'+2y=0 \quad \Rightarrow y''+2\frac{1-x}{x^2-2x}y'+\frac{2}{x^2-2x}y=0$

已知其一解為 $y_1=x^2$

由 Wronskian 可知

$$W(y_1,y_2)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=y_1y_2'-y_1'y_2=C_1e^{-\int a dx}$$

$$\Rightarrow y_2'-\frac{y_1'}{y_1}y_2=\frac{C_1}{y_1}e^{-\int a dx}=\frac{C_1}{y_1}e^{2\int\frac{x-1}{x^2-2x}dx}=\frac{C_1}{y_1}e^{\int(\frac{1}{x}+\frac{1}{x-2})dx}=\frac{C_1}{y_1}e^{\ln|x|+\ln|x-2|}=C_1\frac{x(x-2)}{y_1}$$

又 $y_1=x^2$ 代入後可得

$\therefore\ y_2'-\frac{2}{x}y_2=C_1(1-\frac{2}{x})$ ⟶此為一階線性 ODE

可知積分因子為 $\mu=e^{\int p(x)dx}=e^{-\int\frac{2}{x}dx}=e^{-2\ln|x|}=\frac{1}{x^2}$

$$\frac{1}{x^2}y_2'-\frac{2}{x^3}y_2=C_1(\frac{1}{x^2}-\frac{2}{x^3})\quad\Rightarrow\frac{d}{dx}(\frac{1}{x^2}y_2)=C_1(\frac{1}{x^2}-\frac{2}{x^3})$$

$$\Rightarrow\frac{1}{x^2}y_2=C_1(-\frac{1}{x}+\frac{1}{x^2})+C_2$$

$$\Rightarrow y_2=C_1(-x+1)+C_2x^2$$

$y_1=x^2$ 為其一補解，故可知另一補解為 $y_2=1-x$

3. (1) $y''-4y'+2y=0$

令 $y=e^{\lambda x}$ 代回 ODE 可得

$(\lambda^2-4\lambda+2)e^{\lambda x}=0$

$\Rightarrow\lambda^2-4\lambda+2=0$

$$\Rightarrow\lambda=\frac{4\pm\sqrt{(-4)^2-4\cdot1\cdot2}}{2}=2\pm\sqrt{2}$$

$\therefore\ y=C_1e^{(2+\sqrt{2})x}+C_2e^{(2-\sqrt{2})x}$

(2) $y''-10y'+25y=0$

令 $y=e^{\lambda x}$ 代回 ODE 可得

$(\lambda^2-10\lambda+25)e^{\lambda x}=0$

$\Rightarrow\lambda^2-10\lambda+25=0$

$\Rightarrow(\lambda-5)^2=0$

$\Rightarrow\lambda=5,5$ (重根)

$\therefore\ y=C_1e^{5x}+C_2xe^{5x}$

(3) $y''+2y'+6y=0$

令 $y=e^{\lambda x}$ 代回 ODE 可得

$(\lambda^2+2\lambda+6)e^{\lambda x}=0$

$\Rightarrow\lambda^2+2\lambda+6=0$

$$\Rightarrow \lambda = \frac{-2 \pm \sqrt{2^2 - 4\cdot 1\cdot 6}}{2} = -1 \pm \sqrt{5}\, i$$

$\therefore \quad y = e^{-x}(C_1 \cos\sqrt{5}\, x + C_2 \sin\sqrt{5}\, x)$

(4) $y'' - 2y' + (\pi^2 + 1)y = 0$

令 $y = e^{\lambda x}$ 代回 ODE 可得

$[\lambda^2 - 2\lambda + (\pi^2 + 1)]e^{\lambda x} = 0$

$\Rightarrow \lambda^2 - 2\lambda + (\pi^2 + 1) = 0$

$$\Rightarrow \lambda = \frac{2 \pm \sqrt{(-2)^2 - 4\cdot 1\cdot (\pi^2 + 1)}}{2} = 1 \pm \pi\, i$$

$\therefore \quad y = e^{x}(C_1 \cos\pi\, x + C_2 \sin\pi\, x)$

又 $y(\frac{1}{4}) = 0 \quad \Rightarrow e^{\frac{1}{4}}(C_1 \cos\frac{\pi}{4} + C_2 \sin\frac{\pi}{4}) = 0 \quad \Rightarrow C_1 + C_2 = 0 \quad \Rightarrow C_1 = -C_2 \quad \ldots(1)$

$y'(\frac{1}{4}) = -\pi \sqrt[4]{4e}$

$\Rightarrow e^{\frac{1}{4}}(C_1 \cos\frac{\pi}{4} + C_2 \sin\frac{\pi}{4}) + \pi e^{\frac{1}{4}}(-C_1 \sin\frac{\pi}{4} + C_2 \cos\frac{\pi}{4}) = -\pi \sqrt[4]{4e}$

$\Rightarrow C_1(1-\pi) + C_2(1+\pi) = -2\pi \quad \ldots(2)$

將(1)代入(2)可得 $C_2 = -1$ 代回(1)可得 $C_1 = 1$

$\therefore \quad y = e^{x}(\cos\pi\, x - \sin\pi\, x)$

(5) $4y'' + 4y' + 1y = 0$

令 $y = e^{\lambda x}$ 代回 ODE 可得

$(4\lambda^2 + 4\lambda + 1)e^{\lambda x} = 0$

$\Rightarrow 4\lambda^2 + 4\lambda + 1 = 0$

$\Rightarrow \lambda = -\frac{1}{2}, -\frac{1}{2}$ (重根)

$\therefore \quad y = e^{-\frac{x}{2}}(C_1 + C_2 x)$

又 $y(0) = -2 \quad \Rightarrow C_1 = -2$

$y(2) = e^{-1} \quad \Rightarrow C_2 = \frac{3}{2}$

$\therefore \quad y = e^{-\frac{x}{2}}(-2_1 + \frac{3}{2}x)$

(6) $y''' + 6y'' + 12y' + 8y = 0$

令 $y = e^{\lambda x}$ 代回 ODE 可得

$(\lambda^3 + 6\lambda^2 + 12\lambda + 8)e^{\lambda x} = 0$

$\Rightarrow \lambda^3 + 6\lambda^2 + 12\lambda + 8 = 0$

$\Rightarrow (\lambda + 2)^3 = 0$

$\Rightarrow \lambda = -2,\ -2, -2$ (三重根)

$\therefore\quad y = e^{-2x}(C_1 + C_2 x + C_3 x^2)$