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1. 試解下述一階高次微分方程式：

$$(1) x^2(y')^2 + 4xyy' + 3y^2 = 0$$

$$(2) xy(y')^2 + (x+y)y' + 1 = 0$$

$$(3) (y')^4 - (x+2y+1)(y')^3 + (x+2y+2xy)(y')^2 - 2xyy' = 0$$

$$(4) y = xy' - \frac{1}{y'}$$

2. 試以全微分法求解下述微分方程

$$(1) (2x^2y + 3y^3)dx - (x^3 + 2xy^2)dy = 0$$

$$(2) y' = \frac{y + xy^3(1 + \ln x)}{x}$$

$$(3) (3xe^y + 2y)dx + (x^2e^y + x)dy = 0$$

$$(4) y' = \frac{y + \sqrt{x^2 + y^2}}{x}$$

3. 試以皮卡德(Picard)法求解：

$$y' = 1 + y^2, \quad y(0) = 0$$

參考解答：

1.

$$(1) x^2(y')^2 + 4xyy' + 3y^2 = 0$$

$$\begin{aligned} \text{令 } u = y' \text{ 代回 ODE 可得 } x^2u^2 + 4xyu + 3y^2 &= 0 \\ &\Rightarrow (xu + 3y)(xu + y) = 0 \\ &\Rightarrow xu + 3y = 0 \quad \text{or} \quad xu + y = 0 \end{aligned}$$

$$\begin{aligned} \text{(a) } xu + 3y = 0 \quad &\Rightarrow y' = -\frac{3}{x}y \quad \Rightarrow \frac{1}{y}dy = -\frac{3}{x}dx \\ &\Rightarrow \int \frac{1}{y}dy = -\int \frac{3}{x}dx \\ &\Rightarrow \ln|y| = -3\ln|x| + \ln|C| \\ &\Rightarrow y = \frac{C}{x^3} \end{aligned}$$

$$\begin{aligned} \text{(b) } xu + y = 0 \quad &\Rightarrow y' = -\frac{1}{x}y \quad \Rightarrow \frac{1}{y}dy = -\frac{1}{x}dx \\ &\Rightarrow \int \frac{1}{y}dy = -\int \frac{1}{x}dx \end{aligned}$$

$$\Rightarrow \ln|y| = -\ln|x| + \ln|C|$$

$$\Rightarrow y = \frac{C}{x}$$

$$\therefore \text{此 ODE 的解為 } (y - \frac{C}{x^3})(y - \frac{C}{x}) = 0$$

$$(2) \quad xy(y')^2 + (x+y)y' + 1 = 0$$

$$\text{令 } u = y' \text{ 代回 ODE 可得 } xyu^2 + (x+y)u + 1 = 0$$

$$\Rightarrow (xu+1)(yu+1) = 0$$

$$\Rightarrow xu+1=0 \quad \text{or} \quad yu+1=0$$

$$(a) \quad xu+1=0 \Rightarrow y' = -\frac{1}{x} \Rightarrow y = -\ln|x| + C \Rightarrow x = Ce^{-y}$$

$$(b) \quad yu+1=0 \Rightarrow yy' = -1 \Rightarrow \int y dy = -\int dx$$

$$\Rightarrow \frac{1}{2}y^2 = -x + C$$

$$\therefore \text{此 ODE 的解為 } (x - Ce^{-y})(x + \frac{1}{2}y^2 - C) = 0$$

$$(3) \quad (y')^4 - (x+2y+1)(y')^3 + (x+2y+2xy)(y')^2 - 2xyy' = 0$$

$$\text{令 } u = y' \text{ 代回 ODE 可得 } u^4 - (x+2y+1)u^3 + (x+2y+2xy)u^2 - 2xyu = 0$$

$$\Rightarrow u(u-x)(u-2y)(u-1) = 0$$

$$\Rightarrow u=0, \quad u-x=0, \quad u-2y=0 \quad \text{or} \quad u-1=0$$

$$(a) \quad u=0 \Rightarrow y'=0 \Rightarrow y=C$$

$$(b) \quad u-x=0 \Rightarrow y'=x \Rightarrow y = \frac{1}{2}x^2 + C$$

$$(c) \quad u-2y=0 \Rightarrow y'=2y \Rightarrow \ln|y| = 2x + C \Rightarrow y = Ce^{2x}$$

$$(b) \quad u-1=0 \Rightarrow y'=1 \Rightarrow y = x + C$$

$$\therefore \text{此 ODE 的解為 } (y-C)(y - \frac{1}{2}x^2 - C)(y - Ce^{2x})(y - x - C) = 0$$

$$(4) \quad y = xy' - \frac{1}{y'}$$

$$\text{令 } u = y' \text{ 代回 ODE 可得 } y = xu - \frac{1}{u}$$

$$\text{同時將兩邊對 } x \text{ 微分: } y' = u + xu' + \frac{u'}{u^2} \Rightarrow u = u + xu' + \frac{u'}{u^2}$$

$$\Rightarrow u'(x + \frac{1}{u^2}) = 0$$

$$\Rightarrow u' = 0 \quad \text{或} \quad u^2 = -\frac{1}{x}$$

$$\text{當 } u' = 0 \Rightarrow u = c \Rightarrow y = cx - \frac{1}{c} \quad (\text{通解})$$

$$\text{當 } u^2 = -\frac{1}{x} \Rightarrow uy = xu^2 - 1 \Rightarrow uy = -2 \Rightarrow u^2 y^2 = 4 \Rightarrow y^2 = -4x \quad (\text{奇解})$$

2.

$$(1) (2x^2y + 3y^3)dx - (x^3 + 2xy^2)dy = 0$$

$$\Rightarrow (2x^2ydx - x^3dy) + (3y^3dx - 2xy^2dy) = 0$$

$$\Rightarrow x^2(2ydx - xdy) + y^2(3ydx - 2x^2dy) = 0$$

$$\Rightarrow x^2 \frac{d(x^2y^{-1})}{xy^{-2}} + y^2 \frac{d(x^3y^{-2})}{x^2y^{-3}} = 0$$

$$\Rightarrow xy^2d(x^2y^{-1}) + x^{-2}y^5d(x^3y^{-2}) = 0$$

$$\Rightarrow x^{-6}y^3d(x^2y^{-1}) + x^{-9}y^6d(x^3y^{-2}) = 0 \quad (\text{同乘 } x^{-7}y)$$

$$\Rightarrow -\frac{1}{2}(x^2y^{-1})^{-2} - \frac{1}{2}(x^3y^{-2})^{-2} = C$$

$$(2) y' = \frac{y + xy^3(1 + \ln x)}{x} \Rightarrow xdy - ydx = xy^3(1 + \ln x)dx$$

$$\Rightarrow \frac{x}{y} \frac{xdy - ydx}{y^2} = x^2(1 + \ln x)dx$$

$$\Rightarrow -\frac{x}{y} d\left(\frac{x}{y}\right) = x^2(1 + \ln x)dx$$

$$\Rightarrow -\int \frac{x}{y} d\left(\frac{x}{y}\right) = \int x^2(1 + \ln x)dx$$

$$\Rightarrow -\frac{1}{2}\left(\frac{x}{y}\right)^2 = \frac{1}{3}x^3 + \frac{1}{3}x^2 \ln x - \frac{1}{9}x^3 + C$$

$$\Rightarrow -\frac{1}{2}\left(\frac{x}{y}\right)^2 = \frac{2}{3}x^3 + \frac{1}{3}x^2 \ln x + C$$

$$(3) (3xe^y + 2y)dx + (x^2e^y + x)dy = 0$$

$$\Rightarrow (3xe^y dx + x^2e^y dy) + (2ydx + xdy) = 0$$

$$\Rightarrow \frac{d(x^3e^y)}{x} + \frac{d(x^2y)}{x} = 0$$

$$\Rightarrow d(x^3e^y) + d(x^2y) = 0$$

$$\Rightarrow x^3e^y + x^2y = C$$

$$\begin{aligned}
(4) \quad y' &= \frac{y + \sqrt{x^2 + y^2}}{x} \Rightarrow xdy - ydx = \sqrt{x^2 + y^2} dx \\
&\Rightarrow \frac{xdy - ydx}{x^2} = \frac{1}{x} \sqrt{1 + \left(\frac{y}{x}\right)^2} dx \\
&\Rightarrow \frac{1}{\sqrt{1 + \left(\frac{y}{x}\right)^2}} d\left(\frac{y}{x}\right) = \frac{1}{x} dx \\
&\Rightarrow \int \frac{1}{\sqrt{1 + \left(\frac{y}{x}\right)^2}} d\left(\frac{y}{x}\right) = \int \frac{1}{x} dx
\end{aligned}$$

令 $\frac{y}{x} = \tan \theta \Rightarrow d\left(\frac{y}{x}\right) = \sec^2 \theta d\theta$ 代入上式

可得 $\int \sec \theta d\theta = \int \frac{1}{x} dx \Rightarrow \int \sec \theta \cdot \frac{\sec \theta + \tan \theta}{\sec \theta + \tan \theta} d\theta = \int \frac{1}{x} dx$

令 $u = \sec \theta + \tan \theta \Rightarrow du = (\sec \theta \cdot \tan \theta + \sec^2 \theta) d\theta = \sec \theta (\tan \theta + \sec \theta) d\theta$

$\therefore \int \sec \theta \cdot \frac{\sec \theta + \tan \theta}{\sec \theta + \tan \theta} d\theta = \int \frac{1}{u} du = \int \frac{1}{x} dx$

$\Rightarrow \ln|u| = \ln|x| + C$

$\Rightarrow \ln|\sec \theta + \tan \theta| = \ln|x| + C$

$\Rightarrow \ln\left|\sqrt{1 + \tan^2 \theta} + \tan \theta\right| = \ln|x| + C$

$\Rightarrow \ln\left|\sqrt{1 + \left(\frac{y}{x}\right)^2} + \frac{y}{x}\right| = \ln|x| + C$

3. $y' = 1 + y^2, \quad y(0) = 0$

$y = y_0 + \int_{x_0}^x f(t, y(t)) dt$

$y_1 = y_0 + \int_0^x (1 + y_0^2) dt = 0 + \int_0^x dx = x$

$y_2 = y_0 + \int_0^x (1 + y_1^2) dt = 0 + \int_0^x (1 + x^2) dx = x + \frac{1}{3}x^3$

$$\begin{aligned}
y_3 &= y_0 + \int_0^x (1 + y_2^2) dt = 0 + \int_0^x \left(1 + x^2 + \frac{2}{3}x^4 + \frac{1}{9}x^6\right) dx \\
&= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{1}{63}x^7
\end{aligned}$$

$$\begin{aligned}
y_4 &= y_0 + \int_0^x (1 + y_3^2) dt = 0 + \int_0^x [1 + (x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{1}{63}x^7)^2] dx \\
&= \int_0^x (1 + x^2 + \frac{2}{3}x^4 + \frac{17}{45}x^6 + \frac{38}{315}x^8 + \frac{131}{4725}x^{10} + \dots) dx \\
&= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{38}{2835}x^9 + \frac{131}{51975}x^{11} + \dots
\end{aligned}$$

$$\begin{aligned}
y_5 &= y_0 + \int_0^x (1 + y_4^2) dt = 0 + \int_0^x [1 + (x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{38}{2835}x^9 + \dots)^2] dx \\
&= \int_0^x (1 + x^2 + \frac{2}{3}x^4 + \frac{17}{45}x^6 + \frac{62}{315}x^8 + \frac{1142}{14175}x^{10} + \dots) dx \\
&= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \frac{1142}{155925}x^{11} + \dots
\end{aligned}$$

⋮

$$y_n = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \dots$$

解析解:

$$\begin{aligned}
y' = 1 + y^2 &\Rightarrow \frac{1}{1+y^2} dy = dx \Rightarrow \int \frac{1}{1+y^2} dy = \int dx \\
&\Rightarrow \tan^{-1} y = x + C \Rightarrow y = \tan(x + C)
\end{aligned}$$

$$\text{又 } y(0) = 0 \Rightarrow C = 0$$

$$\therefore y = \tan x$$

$$\text{由級數展開可知 } \sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \dots$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \dots$$

$$y = \tan x = \frac{\sin x}{\cos x} = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + O(x^{11})$$