國立臺灣海洋大學河海工程學系 2021 工程數學(一) 2A 班第三次作業參考解答

系級：____________　　學號：____________　　姓名：____________

試解下述一階線性微分方程式:

1. (1) $xy' + (1+x)y = e^x$

   (2) $(x-2)y' = y + 2(x-2)^3$

   (3) $y' - 2y\cot(2x) = 1 - 2x\cot(2x) - 2\csc(2x)$

   (4) $y' = 2\dfrac{y}{x} + x^2 e^x$

   (5) $y' + y = \sin x$

2. (1) $y' - y = xy^5$

   (2) $y' + \dfrac{3}{x}y = -2xy^{\frac{5}{2}}$

3. (1) $y' - y + e^{-x}y^2 - e^x = 0$

   (2) $y' = xy^2 + (-8x^2 + \dfrac{1}{x})y + 16x^3$， $y(2) = 6$

參考解答:

1.

(1) $xy' + (1+x)y = e^x \quad \Rightarrow y' + \dfrac{1+x}{x}y = \dfrac{1}{x}e^x \longrightarrow$ 此為一階線性 ODE

積分因子為 $\mu = e^{\int p(x)dx} = e^{\int \frac{1+x}{x}dx} = e^{x+\ln x} = xe^x$

同乘積分因子後可得 $xe^x y' + xe^x \dfrac{1+x}{x}y = \dfrac{1}{x}e^x \cdot xe^x$

$$\Rightarrow \frac{d}{dx}(xe^x y) = e^{2x}$$

$$\Rightarrow xe^x y = \int \frac{1}{x}e^{2x}\,dx = \frac{1}{2}e^{2x} + C$$

$$\Rightarrow y = \frac{e^x}{2x} + C\frac{e^{-x}}{x}$$

(2) $(x-2)y' = y + 2(x-2)^3 \quad \Rightarrow y' - \dfrac{1}{x-2}y = 2(x-2)^2 \longrightarrow$ 此為一階線性 ODE

積分因子為 $\mu = e^{\int p(x)dx} = e^{-\int \frac{1}{x-2}dx} = e^{-\ln|x-2|} = \dfrac{1}{x-2}$

同乘積分因子後可得 $\dfrac{1}{x-2}y' - \dfrac{1}{(x-2)^2}y = 2(x-2)$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{x-2}y\right) = 2(x-2)$$

$$\Rightarrow \frac{1}{x-2}y = 2\int (x-2)\,dx = (x-2)^2 + C$$

$$\Rightarrow y = (x-2)^3 + C(x-2)$$

(3) $y' - 2y\cot(2x) = 1 - 2x\cot(2x) - 2\csc(2x)$ ⟶ 此為一階線性 ODE

積分因子為 $\mu = e^{\int p(x)dx} = e^{-2\int \cot(2x)dx} = e^{-\ln|\sin(2x)|} = \dfrac{1}{\sin(2x)}$

同乘積分因子後可得

$$\frac{1}{\sin(2x)}y' - 2\frac{\cos(2x)}{\sin^2(2x)}y = \frac{1}{\sin(2x)} - 2x\frac{\cos(2x)}{\sin^2(2x)} - 2\frac{1}{\sin^2(2x)}$$

$$\Rightarrow \frac{d}{dx}\left[\frac{1}{\sin(2x)}y\right] = \frac{1}{\sin(2x)} - 2x\frac{\cos(2x)}{\sin^2(2x)} - 2\frac{1}{\sin^2(2x)}$$

$$\Rightarrow \frac{1}{\sin(2x)}y = \int\left[\frac{1}{\sin(2x)} - 2x\frac{\cos(2x)}{\sin^2(2x)} - 2\frac{1}{\sin^2(2x)}\right]dx$$

$$\Rightarrow \frac{1}{\sin(2x)}y = \frac{1}{2}(\ln|\sin x| - \ln|\cos x|) + \frac{x}{\sin 2x} - \frac{1}{2}(\ln|\sin x| - \ln|\cos x|) + \cot(2x) + C$$

$$\Rightarrow y = x + \cos(2x) + C\sin(2x)$$

Note:

$$\int \frac{1}{\sin(2x)}\,dx = \frac{1}{2}\int\left(\frac{\cos x}{\sin x} + \frac{\sin x}{\cos x}\right)dx = \frac{1}{2}(\ln|\sin x| - \ln|\cos x|)$$

$$\int x\frac{\cos(2x)}{\sin^2(2x)}\,dx = -\frac{1}{2}\frac{x}{\sin 2x} + \frac{1}{4}(\ln|\sin x| - \ln|\cos x|)$$

$$\int \frac{1}{\sin^2(2x)}\,dx = \int \frac{\sin^2(2x) + \cos^2(2x)}{\sin^2(2x)}\,dx = -\frac{1}{2}\frac{\cos(2x)}{\sin(2x)} = -\frac{1}{2}\cot(2x)$$

(4) $y' = 2\dfrac{y}{x} + x^2e^x \quad \Rightarrow y' - \dfrac{2}{x}y = x^2e^x$ ⟶ 此為一階線性 ODE

積分因子為 $\mu = e^{\int p(x)dx} = e^{-\int \frac{2}{x}dx} = e^{-2\ln|x|} = \dfrac{1}{x^2}$

同乘積分因子後可得 $\dfrac{1}{x^2}y' - \dfrac{2}{x^3}y = e^x$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{x^2}y\right) = e^x$$

$$\Rightarrow \frac{1}{x^2} y = \int e^x \, dx = e^x + C$$

$$\Rightarrow y = x^2 e^x + Cx^2$$

(5) $y' + y = \sin x$ ⟶ 此為一階線性 ODE

積分因子為 $\mu = e^{\int p(x)dx} = e^{\int 1 dx} = e^x$

同乘積分因子後可得 $e^x y' + e^x y = e^x \sin x$

$$\Rightarrow \frac{d}{dx}(e^x y) = e^x \sin x$$

$$\Rightarrow e^x y = \int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + C$$

$$\Rightarrow y = \frac{1}{2}(\sin x - \cos x) + Ce^{-x}$$

2.

(1) $y' - y = xy^5$ ⟶ 此為 Bernoulli ODE

$\Rightarrow y^{-5} y' - y^{-4} = x$

令 $u = y^{-4}$ $\Rightarrow u' = -4y^{-5} y'$ 代回 ODE 可得

$-\frac{1}{4} u' - u = x$ $\Rightarrow u' + 4u = -4x$ ⟶ 此為一階線性 ODE

積分因子為 $\mu = e^{\int p(x)dx} = e^{\int 4 dx} = e^{4x}$

同乘積分因子後可得 $e^{4x} u' + 4e^{4x} u = -4xe^{4x}$

$$\Rightarrow \frac{d}{dx}(e^{4x} u) = -4xe^{4x}$$

$$\Rightarrow e^{4x} u = -4\int xe^{4x} \, dx = e^{4x}(\frac{1}{4} - x) + C$$

$$\Rightarrow u = \frac{1}{4} - x + Ce^{-4x}$$

$$\Rightarrow \frac{1}{y^4} = \frac{1}{4} - x + Ce^{-4x}$$

$$\Rightarrow e^{4x}(y^{-4} + x - \frac{1}{4}) = C$$

(2) $y' + \frac{3}{x} y = -2xy^{\frac{5}{2}}$ ⟶ 此為 Bernoulli ODE

$\Rightarrow y^{-\frac{5}{2}} y' + \frac{3}{x} y^{-\frac{3}{2}} = -2x$

令 $u = y^{-\frac{3}{2}} \quad \Rightarrow u' = -\frac{3}{2} y^{-\frac{5}{2}} y'$ 代回 ODE 可得

$-\frac{2}{3}u' + \frac{3}{x}u = -2x \quad \Rightarrow u' - \frac{9}{2x}u = 3x \longrightarrow$ 此為一階線性 ODE

積分因子為 $\mu = e^{\int p(x)dx} = e^{-\frac{9}{2}\int \frac{1}{x}dx} = e^{-\frac{9}{2}\ln|x|} = x^{-\frac{9}{2}}$

同乘積分因子後可得 $x^{-\frac{9}{2}}u' - \frac{9}{2}x^{-\frac{11}{2}}u = 3x^{-\frac{7}{2}}$

$$\Rightarrow \frac{d}{dx}(x^{-\frac{9}{2}}u) = 3x^{-\frac{7}{2}}$$

$$\Rightarrow x^{-\frac{9}{2}}u = 3\int x^{-\frac{7}{2}}\, dx = -\frac{6}{5}x^{-\frac{5}{2}} + C$$

$$\Rightarrow u = -\frac{6}{5}x^2 + Cx^{\frac{9}{2}}$$

$$\Rightarrow y^{-\frac{3}{2}} = -\frac{6}{5}x^2 + Cx^{\frac{9}{2}}$$

3.

(1) $y' - y + e^{-x}y^2 - e^x = 0 \quad \longrightarrow$ 此為 Riccati ODE

由觀察得一解為 $S = e^x$

令 $y = S + \frac{1}{V} = e^x + \frac{1}{V} \quad \Rightarrow y' = e^x - \frac{V'}{V^2}$ 代回 ODE 可得

$$(e^x - \frac{V'}{V^2}) - (e^x + \frac{1}{V}) + e^{-x}(e^x + \frac{1}{V})^2 - e^x = 0$$

$$\Rightarrow e^x - \frac{V'}{V^2} - e^x - \frac{1}{V} + e^x + \frac{2}{V} + \frac{1}{V^2}e^{-x} - e^x = 0$$

$\Rightarrow V' - V = e^{-x} \longrightarrow$ 此為一階線性 ODE

積分因子為 $\mu = e^{\int p(x)dx} = e^{-\int 1 dx} = e^{-x}$

同乘積分因子後可得 $e^{-x}V' - e^{-x}V = e^{-2x}$

$$\Rightarrow \frac{d}{dx}(e^{-x}V) = e^{-2x}$$

$$\Rightarrow e^{-x}V = \int e^{-2x}\, dx = -\frac{1}{2}e^{-2x} + C$$

$$\Rightarrow V = -\frac{1}{2}e^{-x} + Ce^x$$

$$\therefore\ y = S + V = e^x + \frac{1}{-\frac{1}{2}e^{-x} + Ce^x}$$

(2) $y' = xy^2 + (-8x^2 + \frac{1}{x})y + 16x^3 \longrightarrow$ 此為 Riccati ODE

由觀察得一解為 $S = 4x$

令 $y = S + \frac{1}{V} = 4x + \frac{1}{V} \quad \Rightarrow y' = 4 - \frac{V'}{V^2}$ 代回 ODE 可得

$$4 - \frac{V'}{V^2} = x(4x + \frac{1}{V})^2 + (-8x^2 + \frac{1}{x})(4x + \frac{1}{V}) + 16x^3$$

$$\Rightarrow 4 - \frac{V'}{V^2} = 16x^3 + \frac{8}{V}x^2 + \frac{1}{V^2}x - 32x^3 + 4 - \frac{8x^2}{V} + \frac{1}{Vx} + 16x^3$$

$\Rightarrow V' + \frac{1}{x}V = -x \longrightarrow$ 此為一階線性 ODE

積分因子為 $\mu = e^{\int p(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = x$

同乘積分因子後可得 $xV' + V = -x^2$

$$\Rightarrow \frac{d}{dx}(xV) = -x^2$$

$$\Rightarrow xV = -\int x^2\, dx = -\frac{1}{3}x^3 + C$$

$$\Rightarrow V = -\frac{1}{3}x^2 + C\frac{1}{x}$$

$$\therefore\ y = S + V = 4x + \frac{1}{-\frac{1}{3}x^2 + C\frac{1}{x}}$$

又 $y(2) = 6 \quad \Rightarrow C = \frac{5}{3}$

$$\therefore\ y = 4x + \frac{1}{\frac{5}{3x} - \frac{1}{3}x^2}$$