國立臺灣海洋大學河海工程學系 2021 工程數學(一) 2A 班第二次作業參考解答

系級：________ 學號：________ 姓名：________

1. 試以變數變換法求解 $y' = \dfrac{x+2y+7}{-2x+y-9}$

試以正合法求下述微分方程式:

2. (1) $(2x^3+3y)dx+(3x+y-1)dy=0$

   (2) $x^3 - y\sin x + (\cos x + 2y)y' = 0$

   (3) $(y^2 e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)dy = 0$

   (4) $\dfrac{dy}{dx} = \dfrac{-\cos(xy)+xy\sin(xy)}{-x^2\sin(xy)+2y}$

3. (1) $xdy - ydx - (1-x^2)dx = 0$

   (2) $\sin ydx + \cos ydy = 0$

   (3) $2dx - e^{y-x}dy = 0$

   (4) $(2xy^2+y)dx + (x+2x^2y-x^4y^3)dy = 0$

   (5) $(7x^5y^5 + 2y\sin x + xy\cos x)dx + (6x^6y^4 + 2x\sin x)dy = 0$

參考解答:

1. $(x+5)^2 + 4(x+5)(y+1) - (y+1)^2 = C$

2.

(1) $(2x^3+3y)dx+(3x+y-1)dy=0$

令 $M = 2x^3+3y \quad \Rightarrow \dfrac{\partial M}{\partial y} = 3$

$N = 3x+y-1 \quad \Rightarrow \dfrac{\partial N}{\partial x} = 3$

由判斷式 $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ 可知此為正合微分方程

$\therefore \ M = \dfrac{\partial \phi}{\partial x} = 2x^3+3y \quad \Rightarrow \phi = \dfrac{1}{2}x^4 + 3xy + f(y)$

$N = \dfrac{\partial \phi}{\partial y} = 3x+y-1 \quad \Rightarrow \phi = 3xy + \dfrac{1}{2}y^2 - y + g(x)$

由上兩式可知 $\phi(x,y)=\frac{1}{2}x^4+3xy+\frac{1}{2}y^2-y=C$

(2) $x^3-y\sin x+(\cos x+2y)y'=0 \quad \Rightarrow (x^3-y\sin x)dx+(\cos x+2y)dy=0$

令 $M=x^3-y\sin x \quad \Rightarrow \frac{\partial M}{\partial y}=-\sin x$

$N=\cos x+2y \quad \Rightarrow \frac{\partial N}{\partial x}=-\sin x$

由判斷式 $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ 可知此為正合微分方程

$\therefore \; M=\frac{\partial \phi}{\partial x}=x^3-y\sin x \quad \Rightarrow \phi=\frac{1}{4}x^4+y\cos x+f(y)$

$N=\frac{\partial \phi}{\partial y}=\cos x+2y \quad \Rightarrow \phi=y\cos x+y^2+g(x)$

由上兩式可知 $\phi(x,y)=\frac{1}{4}x^4+y\cos x+y^2=C$

(3) $(y^2e^{xy^2}+4x^3)dx+(2xye^{xy^2}-3y^2)dy=0$

令 $M=y^2e^{xy^2}+4x^3 \quad \Rightarrow \frac{\partial M}{\partial y}=2ye^{xy^2}+y^2e^{xy^2}\cdot 2xy$

$N=2xye^{xy^2}-3y^2 \quad \Rightarrow \frac{\partial N}{\partial x}=2ye^{xy^2}+2xye^{xy^2}\cdot y^2$

由判斷式 $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ 可知此為正合微分方程

$\therefore \; M=\frac{\partial \phi}{\partial x}=y^2e^{xy^2}+4x^3 \quad \Rightarrow \phi=e^{xy^2}+x^4+f(y)$

$N=\frac{\partial \phi}{\partial y}=2xye^{xy^2}-3y^2 \quad \Rightarrow \phi=e^{xy^2}-y^3+g(x)$

由上兩式可知 $\phi(x,y)=e^{xy^2}+x^4-y^3=C$

(4) $\frac{dy}{dx}=\frac{-\cos(xy)+xy\sin(xy)}{-x^2\sin(xy)+2y} \Rightarrow [\cos(xy)-xy\sin(xy)]dx+[-x^2\sin(xy)+2y]dy=0$

令 $M=\cos(xy)-xy\sin(xy) \quad \Rightarrow \frac{\partial M}{\partial y}=-x\sin(xy)-x\sin(xy)-x^2y\cos(xy)$

$N=-x^2\sin(xy)+2y \quad \Rightarrow \frac{\partial N}{\partial x}=-2x\sin(xy)-x^2y\cos(xy)$

由判斷式 $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ 可知此為正合微分方程

$$\therefore \quad M=\frac{\partial \phi}{\partial x}=\cos(xy)-xy\sin(xy)$$

$$\Rightarrow \phi=\frac{1}{y}\sin(xy)-[-\frac{x}{y}\cos(xy)+\frac{1}{y^2}\sin(xy)]\cdot y+f(y)$$

$$\Rightarrow \phi=x\cos(xy)+f(y)$$

$$N=\frac{\partial \phi}{\partial y}=-x^2\sin(xy)+2y \quad \Rightarrow \phi=x\cos(xy)+y^2+g(x)$$

由上兩式可知 $\phi(x,y)=x\cos(xy)+y^2=C$

3.

(1) $xdy-ydx-(1-x^2)dx=0 \Rightarrow (1+y-x^2)dx-xdy=0$

令 $M=1+y-x^2 \quad \Rightarrow \frac{\partial M}{\partial y}=1$

$N=-x \quad \Rightarrow \frac{\partial N}{\partial x}=-1$

由判斷式 $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$ 可知此非正合微分方程

由 $\frac{1}{N}(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x})=-\frac{2}{x}$ 可知 $\mu=\mu(x)$

故可得 $\int\frac{1}{\mu}d\mu=\int\frac{1}{N}(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x})dx=-\int\frac{2}{x}dx \quad \Rightarrow \ln|\mu|=-2\ln|x|$

$$\Rightarrow \mu=\frac{1}{x^2}$$

同乘積分因子後可得 $\frac{1+y-x^2}{x^2}dx-\frac{1}{x}dy=0$ 此為正合微分方程

$$\therefore \quad \overline{M}=\frac{\partial \phi}{\partial x}=\frac{1+y-x^2}{x^2} \quad \Rightarrow \phi=-\frac{1}{x}-\frac{y}{x}-x+f(y)$$

$$\overline{N}=\frac{\partial \phi}{\partial y}=-\frac{1}{x} \quad \Rightarrow \phi=-\frac{y}{x}+g(x)$$

由上兩式可知 $\phi(x,y)=\frac{1}{x}+\frac{y}{x}+x=C$

(2) $\sin ydx+\cos ydy=0$

令 $M=\sin y \quad \Rightarrow \frac{\partial M}{\partial y}=\cos y$

$N=\cos y \quad \Rightarrow \frac{\partial N}{\partial x}=0$

由判斷式 $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ 可知此非正合微分方程

由 $\frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) = 1$ 可知 $\mu = \mu(x)$

故可得 $\int \frac{1}{\mu} d\mu = \int \frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) dx = \int dx \quad \Rightarrow \ln|\mu| = x$

$\Rightarrow \mu = e^x$

同乘積分因子後可得 $e^x \sin y dx + e^x \cos y dy = 0$ 此為正合微分方程

$\therefore \ \overline{M} = \frac{\partial \phi}{\partial x} = e^x \sin y \quad \Rightarrow \phi = e^x \sin y + f(y)$

$\overline{N} = \frac{\partial \phi}{\partial y} = e^x \cos y \quad \Rightarrow \phi = e^x \sin y + g(x)$

由上兩式可知 $\phi(x, y) = e^x \sin y = C$

(3) $2dx - e^{y-x} dy = 0$

令 $M = 2 \quad \Rightarrow \frac{\partial M}{\partial y} = 0$

$N = -e^{y-x} \quad \Rightarrow \frac{\partial N}{\partial x} = e^{y-x}$

由判斷式 $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ 可知此非正合微分方程

由 $\frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) = 1$ 可知 $\mu = \mu(x)$

故可得 $\int \frac{1}{\mu} d\mu = \int \frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) dx = \int dx \quad \Rightarrow \ln|\mu| = x$

$\Rightarrow \mu = e^x$

同乘積分因子後可得 $2e^x dx - e^y dy = 0$ 此為正合微分方程

$\therefore \ \overline{M} = \frac{\partial \phi}{\partial x} = 2e^x \quad \Rightarrow \phi = 2e^x + f(y)$

$\overline{N} = \frac{\partial \phi}{\partial y} = -e^y \quad \Rightarrow \phi = -e^y + g(x)$

由上兩式可知 $\phi(x, y) = 2e^x - e^y = C$

(4) $(2xy^2 + y)dx + (x + 2x^2 y - x^4 y^3)dy = 0$

令 $M = 2xy^2 + y \quad \Rightarrow \frac{\partial M}{\partial y} = 2xy + 1$

$N = x + 2x^2 y - x^4 y^3 \quad \Rightarrow \frac{\partial N}{\partial x} = 1 + 4xy - 4x^3 y^3$

由判斷式 $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ 可知此非正合微分方程

由$\frac{1}{N}(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x})$與$\frac{1}{M}(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y})$ 可知μ非單純為x函數或y函數

$\therefore$ 令$\mu = x^m y^n$

同乘積分因子後可得

$$(2x^{m+1}y^{n+2}+x^m y^{n+1})dx+(x^{m+1}y^n+2x^{m+2}y^{n+1}-x^{m+4}y^{n+3})dy=0$$

此為正合微分方程

$\therefore$ $\overline{M}=2x^{m+1}y^{n+2}+x^m y^{n+1}$ $\Rightarrow \frac{\partial \overline{M}}{\partial y}=2(n+2)x^{m+1}y^{n+1}+(n+1)x^m y^n$

$\overline{N}=x^{m+1}y^n+2x^{m+2}y^{n+1}-x^{m+4}y^{n+3}$

$\Rightarrow \frac{\partial \overline{N}}{\partial x}=(m+1)x^m y^n+2(m+2)x^{m+1}y^{n+1}-(m+4)x^{m+3}y^{n+3}$

又 $\frac{\partial \overline{M}}{\partial y}=\frac{\partial \overline{N}}{\partial x}$ $\Rightarrow m=-4, n=-4$

$\therefore$ $\overline{M}=\frac{\partial \phi}{\partial x}=2x^{-3}y^{-2}+x^{-4}y^{-3}$ $\Rightarrow \phi=-x^{-2}y^{-2}-\frac{1}{3}x^{-3}y^{-3}+f(y)$

$\overline{N}=\frac{\partial \phi}{\partial y}=x^{-3}y^{-4}+2x^{-2}y^{-3}-y^{-1}$ $\Rightarrow \phi=-\frac{1}{3}x^{-3}y^{-3}-x^{-2}y^{-2}-\ln|y|+g(x)$

由上兩式可知 $\phi(x,y)=\frac{1}{3}x^{-3}y^{-3}+x^{-2}y^{-2}+\ln|y|=C$

(5) $(7x^5y^5+2y\sin x+xy\cos x)dx+(6x^6y^4+2x\sin x)dy=0$

令 $M=7x^5y^5+2y\sin x+xy\cos x$ $\Rightarrow \frac{\partial M}{\partial y}=35x^5y^4+2\sin x+x\cos x$

$N=6x^6y^4+2x\sin x$ $\Rightarrow \frac{\partial N}{\partial x}=36x^5y^4+2\sin x+2x\cos x$

由判斷式 $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$ 可知此非正合微分方程

由$\frac{1}{N}(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x})$與$\frac{1}{M}(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y})$ 可知μ非單純為x函數或y函數

$\therefore$ 令$\mu = x^m y^n$

同乘積分因子後可得

$$(7x^{m+5}y^{n+5}+2x^m y^{n+1}\sin x+x^{m+1}y^{n+1}\cos x)dx+(6x^{m+6}y^{n+4}+2x^{m+1}y^n\sin x)dy=0$$

此為正合微分方程

$\therefore$ $\overline{M}=7x^{m+5}y^{n+5}+2x^m y^{n+1}\sin x+x^{m+1}y^{n+1}\cos x$

$\Rightarrow \frac{\partial \overline{M}}{\partial y}=7(n+5)x^{m+5}y^{n+4}+2(n+1)x^m y^n\sin x+(n+1)x^{m+1}y^n\cos x$

$\overline{N}=6x^{m+6}y^{n+4}+2x^{m+1}y^n\sin x$

$\Rightarrow \frac{\partial \overline{N}}{\partial x}=6(m+6)x^{m+5}y^{n+4}+2(m+1)x^m y^n\sin x+2x^{m+1}y^n\cos x$

又 $\dfrac{\partial \overline{M}}{\partial y}=\dfrac{\partial \overline{N}}{\partial x} \quad \Rightarrow m=1, n=1$

$\therefore \quad \overline{M}=\dfrac{\partial \phi}{\partial x}=7x^6y^6+2xy^2\sin x+x^2y^2\cos x \quad \Rightarrow \phi=x^7y^6+x^2y^2\sin x+f(y)$

$\overline{N}=\dfrac{\partial \phi}{\partial y}=6x^7y^5+2x^2y\sin x \quad \Rightarrow \phi=x^7y^6+x^2y^2\sin x+g(x)$

由上兩式可知 $\phi(x,y)=x^7y^6+x^2y^2\sin x=C$