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試求下述微分方程式：

1. (1)  $y(1+x^2)y' = 3x + xy^2$

(2)  $e^{x+y} \frac{dy}{dx} = 2x$

(3)  $(x \ln x)y' = y$

(4)  $y' = x \csc^2 y$

2.  $xy' = y^2 + y$

3.  $y' = \frac{x^3 + y^3}{3xy^2}$

4.  $(y^2 + 2y + 5)dx - dy = 0$

**參考解答：**

1. (1)  $y(1+x^2)y' = 3x + xy^2 \Rightarrow y(1+x^2) \frac{dy}{dx} = x(3+y^2)$   
 $\Rightarrow \frac{y}{3+y^2} dy = \frac{x}{1+x^2} dx$   
 (兩邊積分)  $\Rightarrow \int \frac{y}{3+y^2} dy = \int \frac{x}{1+x^2} dx$   
 $\Rightarrow \frac{1}{2} \ln(y^2 + 3) = \frac{1}{2} \ln(x^2 + 1) + C_1$   
 $\Rightarrow \ln\left(\frac{y^2 + 3}{x^2 + 1}\right) = C_2$   
 $\Rightarrow \frac{y^2 + 3}{x^2 + 1} = e^{C_2} = C$   
 $\Rightarrow y^2 = C(x^2 + 1) - 3$

(2)  $e^{x+y} \frac{dy}{dx} = 2x \Rightarrow e^y dy = 2xe^{-x} dx$   
 (兩邊積分)  $\Rightarrow \int e^y dy = \int 2xe^{-x} dx$   
 $\Rightarrow e^y = -2(1+x)e^{-x} + C$

(3)  $(x \ln x)y' = y \Rightarrow \frac{1}{y} dy = \frac{1}{x \ln x} dx$   
 (兩邊積分)  $\Rightarrow \int \frac{1}{y} dy = \int \frac{1}{x \ln x} dx$

$$\begin{aligned} &\Rightarrow \int \frac{1}{y} dy = \int \frac{1}{\ln x} d(\ln x) \\ &\Rightarrow \ln y = \ln(\ln x) + \ln C \\ &\Rightarrow y = C \ln x \end{aligned}$$

$$(4) \quad y' = x \csc^2 y \quad \Rightarrow \frac{dy}{dx} = x \csc^2 y = \frac{x}{\sin^2 y}$$

$$\Rightarrow \sin^2 y dy = x dx$$

$$\begin{aligned} \text{(兩邊積分)} \quad &\Rightarrow \int \sin^2 y dy = \int x dx \\ &\Rightarrow \int \frac{1 - \cos 2y}{2} dy = \frac{1}{2} x^2 + C_1 \\ &\Rightarrow \frac{1}{2} y - \frac{1}{4} \sin 2y = \frac{1}{2} x^2 + C_1 \\ &\Rightarrow 2y - \sin 2y = 2x^2 + C \end{aligned}$$

$$2. \quad xy' = y^2 + y \quad \Rightarrow \frac{1}{y(y+1)} dy = \frac{1}{x} dx$$

$$\begin{aligned} \text{(兩邊積分)} \quad &\Rightarrow \int \frac{1}{y(y+1)} dy = \int \frac{1}{x} dx \\ &\Rightarrow \int \left( \frac{1}{y} - \frac{1}{y+1} \right) dy = \int \frac{1}{x} dx \\ &\Rightarrow \ln|y| - \ln|y+1| = \ln|x| + \ln|C| \\ &\Rightarrow \frac{y}{y+1} = C_1 x \\ &\Rightarrow 1 + \frac{1}{y} = \frac{C}{x} \\ &\Rightarrow \frac{1}{y} = \frac{C-x}{x} \\ &\Rightarrow y = \frac{x}{C-x} \end{aligned}$$

$$3. \quad y' = \frac{x^3 + y^3}{3xy^2} \quad \rightarrow \quad \text{齊次型 ODE}$$

$$\Rightarrow y' = \frac{1}{3} \left( \frac{x^2}{y^2} + \frac{y}{x} \right)$$

$$\text{令 } u = \frac{y}{x} \quad \Rightarrow y = ux \quad \Rightarrow y' = u'x + u$$

$$\begin{aligned}
\therefore y' = \frac{1}{3} \left( \frac{x^2}{y^2} + \frac{y}{x} \right) &\Rightarrow u'x + u = \frac{1}{3} \left( \frac{1}{u^2} + u \right) \\
&\Rightarrow u'x = \frac{1 - 2u^3}{3u^2} \\
&\Rightarrow \frac{3u^2}{1 - 2u^3} du = \frac{1}{x} dx \\
(\text{兩邊積分}) &\Rightarrow \int \frac{3u^2}{1 - 2u^3} du = \int \frac{1}{x} dx \\
&\Rightarrow -\frac{1}{2} \int \frac{1}{1 - 2u^3} d(1 - 2u^3) = \int \frac{1}{x} dx \\
&\Rightarrow -\frac{1}{2} \ln|1 - 2u^3| = \ln|x| + \ln|C_1| \\
&\Rightarrow \ln|1 - 2u^3| + 2\ln|x| = \ln|C| \\
&\Rightarrow (1 - 2u^3) \cdot x^2 = C \\
&\Rightarrow x^3 - 2y^3 = Cx
\end{aligned}$$

$$4. (y^2 + 2y + 5)dx - dy = 0 \Rightarrow \frac{1}{y^2 + 2y + 5} dy = dx$$

$$(\text{兩邊積分}) \Rightarrow \int \frac{1}{y^2 + 2y + 5} dy = \int dx$$

$$\Rightarrow \int \frac{1}{(y+1)^2 + 4} dy = \int dx$$

$$\text{令 } u = y + 1 \Rightarrow du = dy$$

$$\int \frac{1}{(y+1)^2 + 4} dy = \int dx \Rightarrow \int \frac{1}{u^2 + 4} du = \int dx$$

$$\text{令 } u = 2 \tan \theta \Rightarrow du = 2 \sec^2 \theta d\theta$$

$$\Rightarrow \int \frac{1}{u^2 + 4} du = \int dx \Rightarrow \frac{1}{2} \int \frac{\sec^2 \theta}{\tan^2 \theta + 1} d\theta = \int dx$$

$$\Rightarrow \frac{1}{2} \int d\theta = \int dx$$

$$\Rightarrow \theta = 2x + C$$

$$\Rightarrow \tan^{-1} \frac{u}{2} = 2x + C$$

$$\Rightarrow y + 1 = 2 \tan(2x + C)$$