國立臺灣海洋大學河海工程學系 2021 工程數學(一)第二次大考參考解答

系級：______________　　學號：____________　　姓名：___________

1. 若已知一 2 階 ODE

$$ax^2y''(x)+bxy'(x)+cy(x)=-3x^2-8x-3$$

且滿足 $y'(1)=y(1)=0$ 與其特解 $y_p(x)=x^2+2x+1$

(1) 請問 a,b,c 分別為何? (6%)

(2) 請問兩個補解 $y_1(x)$ 與 $y_2(x)$ 為何? (6%)

(3) 請問全解 $y(x)$ 為何? (2%)

2. 考慮下述三條微分方程式

(a) $y''(t)-3y'(t)-3y(t)=0$　(b) $y''(t)+4y(t)=0$　(c) $y''(t)+8y'(t)+15y(t)=0$

試問: (1) 當 $t\to\infty$，何者會產生週期性振動的解? (4%)

(2) 當 $t\to\infty$，何者的解會衰減到零? (4%)

(3) 當 $t\to\infty$，何者會產生無窮大的解? (4%)

3. 已知二階 ODE

$$y''(x)+y'(x)-6y(x)=e^{2x}$$

(1) 試以待定係數法(Undeterminated coefficient method)求特解。 (8%)

(2) 試以參數變異法(Variation parameter method)求特解。 (8%)

4. 試求微分方程式 $x^2y''+5xy'+4y=\dfrac{2\ln x}{x^2}$ 之通解。 (10%)

5. 試求: (1) $(3x+2)^2y''+(9x+6)y'-36y=27x+9$ (8%)

(2) $x(1-x)y''+2(1-2x)y'-2y=0$ (8%)

(3) $y''+e^{3y}(y')^3=0$ (7%)

6. (1) 試求微分方程 $y''(t)+9\omega_0^2y(t)=\cos(3\omega t)$ 之兩補解 $y_1(t)$ 與 $y_2(t)$ 與其 Wronskian，即 $W(y_1,y_2)=?$。 (5%)

(2) 當 $\omega\neq\omega_0$ 時，此微分方程之特解為何? (5%)

(3) 當 $\omega=\omega_0$ 時，此微分方程之特解為何? (5%)

7. 已知 $y(x)=e^{ax}$ 為方程式 $xy''+2(1-x)y'+(x-2)y=xe^x$ 之一補解

(1) 試問: $a=?$ (2%)

(2) 試求此方程式之通解。 (8%)

參考解答:

1. 若已知一 2 階 ODE

$$ax^2 y''(x) + bxy'(x) + cy(x) = -3x^2 - 8x - 3$$

且滿足 $y'(1) = y(1) = 0$ **與其特解** $y_p(x) = x^2 + 2x + 1$

(1) 請問 a, b, c **分別為何? (6%)**

(2) 請問兩個補解 $y_1(x)$ **與** $y_2(x)$ **為何? (6%)**

(3) 請問全解 $y(x)$ **為何? (2%)**

(1) $2ax^2 + bx(2x+2) + c(x^2 + 2x + 1) = -3x^2 - 8x - 3$

$\Rightarrow a = 1, b = -1, c = -3$

$\therefore x^2 y''(x) - xy'(x) - 3y(x) = -3x^2 - 8x - 3$

(2) $y_1(x) = x^3,\ y_2(x) = x^{-1}$

(3) $y(x) = -2x^3 - 2x^{-1} + x^2 + 2x + 1$

2. 考慮下述三條微分方程式

(a) $y''(t) - 3y'(t) - 3y(t) = 0$ **(b)** $y''(t) + 4y(t) = 0$ **(c)** $y''(t) + 8y'(t) + 15y(t) = 0$

試問: (1) 當 $t \to \infty$ **,何者會產生週期性振動的解? (4%)**

(2) 當 $t \to \infty$ **,何者的解會衰減到零? (4%)**

(3) 當 $t \to \infty$ **,何者會產生無窮大的解? (4%)**

(a) $y(t) = c_1 e^{(\frac{3}{2} - \frac{\sqrt{21}}{2})t} + c_2 e^{(\frac{3}{2} + \frac{\sqrt{21}}{2})t}$

(b) $y(t) = c_1 \cos 2t + c_2 \sin 2t$

(a) $y(t) = c_1 e^{-3t} + c_2 e^{-5t}$

(1) 當 $t \to \infty$,(b)的解會產生週期性振動

(2) 當 $t \to \infty$,(c)的解會衰減到零

(3) 當 $t \to \infty$,(a)的解變成無窮大

3. 已知二階 ODE

$$y''(x) + y'(x) - 6y(x) = e^{2x}$$

(1) 試以待定係數法(Undeterminated coefficient method)求特解。 (8%)

(2) 試以參數變異法(Variation parameter method)求特解。 (8%)

(1) $y''(x) + y'(x) - 6y(x) = e^{2x}$

令 $y = e^{\lambda x} \Rightarrow (\lambda^2 + \lambda - 6)e^{\lambda x} = 0 \quad \Rightarrow \lambda = 2$ or $\lambda = -3$

$\therefore\ y_h = c_1 e^{2x} + c_2 e^{-3x}$

令 $y_p = axe^{2x} \Rightarrow y_p' = a(1 + 2x)e^{2x}$

$$\Rightarrow y_p'' = a(4x+4)e^{2x}$$ 代回 ODE 可得

$$a(4x+4)e^{2x} + a(1+2x)e^{2x} - 6axe^{2x} = e^{2x} \quad \Rightarrow a = \frac{1}{5}$$

$\therefore$ 特解 $y_p = \frac{1}{5}xe^{2x}$， 通解 $y = y_h + y_p = c_1 e^{2x} + c_2 e^{-4x} + \frac{1}{5}xe^{2x}$

(2) 使用參數變異法來求其特解

令其特解 $y_p(x) = u_1 e^{2x} + u_2 e^{-4x}$ 代回 ODE 後可得

$$u_1' = \frac{\begin{vmatrix} 0 & e^{-3x} \\ e^{2x} & -3e^{-3x} \end{vmatrix}}{\begin{vmatrix} e^{2x} & e^{-3x} \\ 2e^{2x} & -3e^{-3x} \end{vmatrix}} = \frac{-e^{-x}}{-5e^{-x}} = \frac{1}{5} \quad \Rightarrow u_1 = \frac{1}{5}x$$

$$u_2' = \frac{\begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & e^{2x} \end{vmatrix}}{\begin{vmatrix} e^{2x} & e^{-3x} \\ 2e^{2x} & -3e^{-3x} \end{vmatrix}} = \frac{e^{4x}}{-5e^{-x}} = -\frac{1}{5}e^{5x} \quad \Rightarrow u_2 = -\frac{1}{25}e^{5x}$$

$$y_p(x) = \frac{1}{5}x \cdot e^{2x} - \frac{1}{25}e^{5x} \cdot e^{-3x} = \frac{1}{5}xe^{2x} - \frac{1}{25}e^{2x}$$ （e^{2x} 為補解，可寫可不寫）

4. 試求微分方程式 $x^2 y'' + 5xy' + 4y = \frac{2\ln x}{x^2}$ 之通解。 (10%)

$$x^2 y'' + 5xy' + 4y = \frac{2\ln x}{x^2}$$

令 $t = \ln x \quad \Rightarrow x = e^t$

$$y'(x) = \frac{dy(x)}{dx} = \frac{dt}{dx}\frac{dY(t)}{dt} = \frac{1}{x}Y'$$

$$y''(x) = \frac{dy'(x)}{dx} = -\frac{1}{x^2}Y' + \frac{1}{x}\frac{dt}{dx}\frac{dY'}{dt} = -\frac{1}{x^2}Y' + \frac{1}{x^2}Y'' = \frac{1}{x^2}(Y'' - Y')$$

代回 ODE 可得: $Y'' + 4Y' + 4Y = 2te^{-2t}$

$\therefore \ Y_h(t) = c_1 e^{-2t} + c_2 te^{-2t}$

令 $Y_p = (at+b)\cdot t^2 e^{-2t} \quad \Rightarrow Y_p' = (-2at^3 - 2bt^2 + 3at^2 + 2bt)\cdot e^{-2t}$

$$\Rightarrow Y_p'' = (4at^3 + 4bt^2 - 12at^2 - 8bt + 6at + 2b)\cdot e^{-2t}$$

代回 ODE 可得: $(4at^3 + 4bt^2 - 12at^2 - 8bt + 6at + 2b)\cdot e^{-2t}$

$$+4(-2at^3 - 2bt^2 + 3at^2 + 2bt)\cdot e^{-2t} + 4(at+b)\cdot t^2 e^{-t} = 2te^{-t}$$

$$\Rightarrow (6at+2b)\cdot e^{-2t} = 2te^{-2t}$$

$$\Rightarrow a=\frac{1}{3},\ \ b=0 \quad \Rightarrow Y_p = \frac{1}{3}t^3 e^{-t}$$

$$\therefore\ \ Y(t) = Y_h(t)+Y_p(t) = c_1 e^{-2t} + c_2 te^{-2t} + \frac{1}{3}t^3 e^{-2t}$$

$$\Rightarrow y(x) = c_1 x^{-2} + c_2 x^{-2}\cdot \ln x + \frac{1}{3}x^{-2}\cdot(\ln x)^3$$

5. 試求: (1) $(3x+2)^2 y'' + (9x+6)y' - 36y = 27x+9$ **(8%)**

(2) $x(1-x)y'' + 2(1-2x)y' - 2y = 0$ **(8%)**

(3) $y'' + e^{3y}(y')^3 = 0$ **(7%)**

(1) $(3x+2)^2 y'' + (9x+6)y' - 36y = 27x+9$

令 $t = 3x+2 \quad \Rightarrow \dfrac{dy(x)}{dx} = \dfrac{dt}{dx}\cdot\dfrac{dY(t)}{dt} = 3Y'(t)$

$$\Rightarrow \frac{d^2y(x)}{dx^2} = \frac{d}{dx}\left(\frac{dy(x)}{dx}\right) = 3\frac{dt}{dx}\cdot\frac{dY'(t)}{dt} = 9Y''(t)$$

$(3x+2)^2 y'' + (9x+6)y' - 36y = 6x+2 \quad \Rightarrow 9t^2Y'' + 9tY' - 36Y = 9t-9$

$$\Rightarrow t^2Y'' + tY' - 4Y = t-1$$

令 $z = \ln t \quad \Rightarrow t = e^z \quad \Rightarrow \dfrac{dY(t)}{dt} = \dfrac{dz}{dt}\cdot\dfrac{dG(z)}{dz} = \dfrac{1}{t}G'(z)$

$$\Rightarrow \frac{d^2Y(t)}{dt^2} = -\frac{1}{t^2}G'(z) + \frac{1}{t}\frac{dz}{dt}\cdot\frac{dG'(z)}{dz} = \frac{1}{t^2}[G''(z) - G'(z)]$$

$t^2Y'' + tY' - 4Y = t-1 \quad \Rightarrow G'' - 4G = e^z - 1$

令 $G = e^{\lambda z} \quad \Rightarrow \lambda^2 - 4 = 0 \quad \Rightarrow \lambda = -2,\, 2$

$\therefore\ \ G_h(z) = c_1 e^{-2z} + c_2 e^{2z}$

令 $G_p = ae^z + b$ 代回 ODE 可得 $a = -\dfrac{1}{3},\ \ b = \dfrac{1}{4}$

$$\therefore\ \ G(z) = G_h(z) + G_p(z) = c_1 e^{-2z} + c_2 e^{2z} - \frac{1}{3}e^z + \frac{1}{4}$$

$$\Rightarrow Y(t) = c_1 t^{-2} + c_2 t^2 - \frac{1}{3}t + \frac{1}{4}$$

$$\Rightarrow y(x) = c_1(3x+2)^{-2} + c_2(3x+2)^2 - x - \frac{1}{3} - \frac{5}{12}$$

(2) $x(1-x)y'' + 2(1-2x)y' - 2y = 0$

令 $a_2 = x(1-x)$, $a_1 = 2(1-2x)$, $a_0 = -2$

由判斷式: $a_2'' - a_1' + a_0 = 0$ 可知此為正合式

$$x(1-x)y''+2(1-2x)y'-2y=\frac{d}{dx}[b_1(x)y'+b_0(x)y]$$
$$\Rightarrow b_1(x)y''+[b_1'(x)+b_0(x)]+b_0'(x)y=x(1-x)y''+2(1-2x)y'-2y$$
$$\Rightarrow b_1=x(1-x),\ b_0=2-2x$$

$$\therefore\ x(1-x)y''+2(1-2x)y'-2y=\frac{d}{dx}[x(1-x)y'+(1-2x)y]=0$$

$\Rightarrow\ x(1-x)y'+(1-2x)y=c_1$　　此為一階線性 ODE

$$\Rightarrow\ y'+\frac{1-2x}{x(1-x)}y=c_1\frac{1}{x(1-x)}$$

積分因子: $\mu=e^{\int\frac{1-2x}{x(1-x)}dx}=e^{\int(\frac{1}{x}+\frac{1}{x-1})dx}=x(x-1)$

同乘積分因子: $x(x-1)y'+(2x-1)y=-c_1$

$$\Rightarrow\frac{d}{dx}[x(x-1)y]=-c_1$$
$$\Rightarrow x(x-1)y=-c_1x+c_2$$
$$\Rightarrow y=-c_1\frac{1}{x-1}+c_2\frac{1}{x(x-1)}$$

(3) $y''+e^{3y}(y')^3=0$ 此 ODE 缺 x

令 $y'=u$ $\Rightarrow y''=\frac{d(y')}{dx}=\frac{dy}{dx}\frac{du}{dy}=u\frac{du}{dy}$ 代回 ODE 可得

$u\frac{du}{dy}+e^{3y}u^3=0$ $\Rightarrow u^{-2}du=-e^{3y}dy$

$$\Rightarrow\int u^{-2}du=\int -e^{3y}dy$$
$$\Rightarrow\frac{1}{u}=\frac{1}{3}e^{3y}+C_1$$
$$\Rightarrow\frac{dx}{dy}=\frac{1}{3}e^{3y}+C_2$$
$$\Rightarrow x=\frac{1}{9}e^{3y}+C_1y+C_2$$

6. (1) 試求微分方程 $y''(t)+9\omega_0^2 y(t)=\cos(3\omega t)$ **之兩補解** $y_1(t)$ **與** $y_2(t)$ **與其 Wronskian，即** $W(y_1,y_2)=?$ **。 (5%)**

(2) 當 $\omega\neq\omega_0$ **時，此微分方程之特解為何? (5%)**

(3) 當 $\omega=\omega_0$ **時，此微分方程之特解為何? (5%)**

(1) $y''(t)+9\omega_0^2 y(t)=0 \Rightarrow y_h(t)=c_1\cos 3\omega_0 t+c_2\sin 3\omega_0 t$

$$W(y_1,y_2)=\begin{vmatrix}\cos 3\omega_0 t & \sin 3\omega_0 t\\ -3\omega_0\sin 2\omega_0 t & 3\omega_0\cos 3\omega_0 t\end{vmatrix}=3\omega_0$$

(2) 當 $\omega\neq\omega_0$ 時

$$y_p=a\cos(3\omega t)+b\sin(3\omega t)\Rightarrow y_p'=-3\omega a\sin(3\omega t)+3\omega b\cos(3\omega t)$$

$$\Rightarrow y_p''=-9\omega^2 a\cos(3\omega t)-9\omega^2 b\sin(3\omega t)$$

$\therefore$ 代回 ODE 並比較係數後可得 $a=\dfrac{1}{9(\omega_0^2-\omega^2)}$, $b=0$

$$\therefore\quad y_p=\frac{1}{9(\omega_0^2-\omega^2)}\cos(3\omega t)$$

(3) 當 $\omega=\omega_0$ 時

$$\lim_{\omega\to\omega_0}\frac{1}{9(\omega_0^2-\omega^2)}[\cos(3\omega t)-\cos(3\omega_0 t)]$$

$$=\lim_{\omega\to\omega_0}\frac{-3t\sin(3\omega t)}{9(-2\omega)}$$

$$=\frac{t\sin(3\omega_0 t)}{6\omega_0}$$

$$\therefore\quad y_p=\frac{t\sin(3\omega_0 t)}{6\omega_0}$$

7. 已知 $y(x)=e^{ax}$ **為方程式** $xy''+2(1-x)y'+(x-2)y=xe^x$ **之一補解**

(1) 試問: $a=?$ **(2%)**

(2) 試求此方程式之通解。 (8%)

(1) $a=1$

(2) $xy''+2(1-x)y'+(x-2)y=xe^x$ 已知一補解 $y_1=e^x$

令另一補解 $y_2=ve^x \Rightarrow y_2'=v'e^x+ve^x$

$$\Rightarrow y_2''=v''e^x+2v'e^x+ve^x$$

帶入 ODE: $xy''+2(1-x)y'+(x-2)y=0$

可得 $x(v''e^x + 2v'e^x + ve^x) + 2(1-x)(v'e^x + ve^x) + (x-2)ve^x = 0$

$\Rightarrow xv''e^x + 2v'e^x = 0$

令 $z = v' \Rightarrow z' + \dfrac{2}{x} z = 0 \Rightarrow z = e^{-2\ln x} = \dfrac{1}{x^2}$

$$\Rightarrow v = -\frac{1}{x}$$

$\therefore$ 另一補解 $y_2 = \dfrac{1}{x} e^x$，且 $y_h = c_1 e^x + c_2 \dfrac{1}{x} e^x$

由參數變異法求特解

令其特解 $y_p(x) = u_1 e^x + u_2 \dfrac{1}{x} e^x$ 代回 ODE 後可得

$$u_1' = \frac{\begin{vmatrix} 0 & x^{-1}e^x \\ e^x & (-x^{-2}+x^{-1})e^x \end{vmatrix}}{\begin{vmatrix} e^x & x^{-1}e^x \\ e^x & (-x^{-2}+x^{-1})e^x \end{vmatrix}} = \frac{-x^{-1}e^{2x}}{-x^{-2}e^{2x}} = x \quad \Rightarrow u_1 = \frac{1}{2}x^2$$

$$u_2' = \frac{\begin{vmatrix} e^x & 0 \\ e^x & e^x \end{vmatrix}}{\begin{vmatrix} e^x & x^{-1}e^x \\ e^x & (-x^{-2}+x^{-1})e^x \end{vmatrix}} = \frac{e^{2x}}{-x^{-2}e^{2x}} = -x^2 \quad \Rightarrow u_2 = -\frac{1}{3}x^3$$

$\therefore \quad y_p(x) = \dfrac{1}{2}x^2 e^x - \dfrac{1}{3}x^2 e^x = \dfrac{1}{6}x^2 e^x$

$y = y_h + y_p = c_1 e^x + c_2 \dfrac{1}{x} e^x + \dfrac{1}{6} x^2 e^x$