國立臺灣海洋大學河海工程學系 2021 工程數學(一) 第一次大考參考解答

系級：______________ 學號：____________ 姓名：___________

1. 試以分離變數法求解下述微分方程式

   (1) $x\sin y\,dx+(x^2+1)\cos y\,dy=0$，$y(1)=\dfrac{\pi}{2}$ (7%)

   (2) $(x^2-xy+y^2)\,dx-xy\,dy=0$ (7%)

   (3) $(x+2y-4)\,dx-(2x+y-5)\,dy=0$ (7%)

   (先轉換成齊次型 ODE，再用變數變換法求解)

2. 已知微分方程式為 $(-xy\sin x+2y\cos x)\,dx+2x\cos x\,dy=0$，試問此微分方程式為正合(exact)或非正合? (2%) 並以正合法求解。(7%)
   (若正合，直接求解；若非正合，先求出積分因子，再求解)

3. 已知微分方程式為 $y^2-(1-2xy)y'=0$

   (1) 此微分方程式為線性或非線性? (2%) 並以一階線性法求解。(7%)
   (若為線性，直接求解；若非線性，則轉換成線性，再求解)

   (2) 此微分方程式為正合(exact)或非正合? (2%) 並以正合法求解。(7%)
   (若正合，直接求解；若非正合，先求出積分因子，再求解)

4. 已知微分方程式為 $x^2\dfrac{dy}{dx}+xy+x^2y^2=1$

   (1) 此為何種類型之微分方程式? (Clairaut、Bernoulli 或是 Riccati) (2%)

   (2) 此為線性或非線性? (2%)

   (3) 試求此微分方程式之解 $y(x)=?$ (7%)

5. 已知微分方程式為 $y=xy'+(y')^2$

   (1) 此為何種類型之微分方程式? (Clairaut、Bernoulli 或是 Riccati) (2%)

   (2) 此為線性或非線性? (2%)

   (3) 試求此微分方程式之解 $y(x)=?$ (8%)

6. 試解下列各微分方程

   (1) $(x+1)y'+y=\ln x$，$y(1)=-1$ (7%) (110 成大土木)

   (2) $(6x^6-x^2y)+(-x^3+x^2y^2)y'=0$，$y(0)=3$ (7%) (110 台大土木)

   (3) $2(y')^2-(2y^2+x)y'+xy^2=0$ (7%)

7. 試以皮卡德法(Picard method)求解 $y'=2y^2$，$y(0)=1$ (取至 y_2 項) (8%)

<參考解答>

1. 試以分離變數法求解下述微分方程式

(1) $x\sin y\,dx+(x^2+1)\cos y\,dy=0$，$y(1)=\dfrac{\pi}{2}$ **(7%)**

(2) $(x^2-xy+y^2)\,dx-xy\,dy=0$ **(7%)**

(3) $(x+2y-4)\,dx-(2x+y-5)\,dy=0$ **(7%)**

(1) $x\sin y\,dx+(x^2+1)\cos y\,dy=0 \quad \Rightarrow \dfrac{\cos y}{\sin y}dy=-\dfrac{x}{x^2+1}dx$

(兩邊積分) $\Rightarrow \displaystyle\int\frac{\cos y}{\sin y}dy=-\int\frac{x}{x^2+1}dx$

$$\Rightarrow \int\frac{1}{\sin y}d(\sin y)=-\frac{1}{2}\int\frac{1}{x^2+1}d(x^2+1)$$

$$\Rightarrow \ln\left|\sin y\right|=-\frac{1}{2}\ln(x^2+1)+\ln C$$

$$\Rightarrow (x^2+1)^{\frac{1}{2}}\cdot\sin y=C$$

又 $y(1)=\dfrac{\pi}{2} \quad \Rightarrow C=\sqrt{2}$

$\therefore\ (x^2+1)^{\frac{1}{2}}\cdot\sin y=\sqrt{2}$

(2) $(x^2-xy+y^2)\,dx-xy\,dy=0 \quad \Rightarrow \dfrac{dy}{dx}=\dfrac{x^2-xy+y^2}{xy}$ ⟶ 齊次型 ODE

$$\Rightarrow \frac{dy}{dx}=\frac{x}{y}-1+\frac{y}{x}$$

令 $u=\dfrac{y}{x} \quad \Rightarrow y=ux \quad \Rightarrow \dfrac{dy}{dx}=x\dfrac{du}{dx}+u$ 代回 ODE 可得

$x\dfrac{du}{dx}+u=\dfrac{1}{u}-1+u \quad \Rightarrow x\dfrac{du}{dx}=\dfrac{1-u}{u}$

$$\Rightarrow \frac{u}{u-1}du=-\frac{1}{x}dx$$

(兩邊積分) $\Rightarrow \displaystyle\int\frac{u}{u-1}du=-\int\frac{1}{x}dx$

$$\Rightarrow u+\ln\left|u-1\right|=-\ln\left|x\right|+\ln\left|C\right|$$

$$\Rightarrow u=-\ln\left|u-1\right|-\ln\left|x\right|+\ln\left|C\right|$$

$$\Rightarrow u = \ln\left|\frac{C}{x(u-1)}\right|$$

$$\Rightarrow x(u-1)\cdot e^{u} = C$$

$$\Rightarrow (y-x)\cdot e^{\frac{y}{x}} = C$$

(3) $(x+2y-4)\,dx-(2x+y-5)\,dy=0$

$$\Rightarrow \frac{dy}{dx}=\frac{x+2y-4}{2x+y-5}$$

令 $x=u+a \quad \Rightarrow dx=du$

$y=v+b \quad \Rightarrow dy=dv$ 代回 ODE 可得

$$\frac{dv}{du}=\frac{u+2v+(a+2b-4)}{2u+v+(2a+b-5)} \qquad \Rightarrow \begin{cases} a+2b-4=0 \\ 2a+b-5=0 \end{cases} \qquad \Rightarrow \begin{cases} a=2 \\ b=1 \end{cases}$$

$\therefore\ \dfrac{dv}{du}=\dfrac{u+2v}{2u+v}$ ⟶ 齊次型 ODE

$$\Rightarrow \frac{dv}{du}=\frac{1+2\frac{v}{u}}{2+\frac{v}{u}}$$

令 $t=\dfrac{v}{u} \quad \Rightarrow v=ut \quad \Rightarrow \dfrac{dv}{du}=t+u\dfrac{dt}{du}$ 代回 ODE 可得

$$t+u\frac{dt}{du}=\frac{1+2t}{2+t} \qquad \Rightarrow u\frac{dt}{du}=\frac{1-t^2}{2+t}$$

$$\Rightarrow \frac{2+t}{1-t^2}\,dt=\frac{1}{u}\,du$$

(兩邊積分) $\Rightarrow \dfrac{1}{2}\displaystyle\int\left(\frac{3}{1-t}+\frac{1}{1+t}\right)dt=\int\frac{1}{u}\,du$

$$\Rightarrow \frac{1}{2}(-3\ln|1-t|+\ln|1+t|)=\ln|u|+\ln C_1$$

$$\Rightarrow (-3\ln|1-t|+\ln|1+t|)=2\ln|u|+2\ln C_1$$

$$\Rightarrow |1-t|^{-3}\cdot|1+t|=C\cdot u^2$$

$$\Rightarrow \left|1+\frac{v}{u}\right|=C\cdot u^2\cdot\left|1-\frac{v}{u}\right|^3$$

$$\Rightarrow |u+v|=C\cdot|u-v|^3$$

$$\Rightarrow |x+y-3|=C\cdot|x-y-1|^3$$

2. 已知微分方程式為 $(-xy\sin x+2y\cos x)\,dx+2x\cos x\,dy=0$，試問此微分方程式為正合(exact)或非正合? (2%) 並以正合法求解。(7%)
(若正合，直接求解；若非正合，先求出積分因子，再求解)

令 $M=-xy\sin x+2y\cos x \quad \Rightarrow \dfrac{\partial M}{\partial y}=-x\sin x+2\cos x$

$N=2x\cos x \quad \Rightarrow \dfrac{\partial N}{\partial x}=2\cos x-2x\sin x$

由判斷式 $\dfrac{\partial M}{\partial y}\neq\dfrac{\partial N}{\partial x}$ 可知，此為非正合 ODE

由 $\dfrac{1}{N}(\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x})=\dfrac{\sin x}{2\cos x}$ 可知 μ 為 $\mu(x)$

$$\therefore\ \frac{1}{\mu}d\mu=\frac{1}{N}(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x})dx \quad \Rightarrow \int\frac{1}{\mu}d\mu=\int\frac{1}{N}(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x})dx$$

$$\Rightarrow \ln|\mu|=\frac{1}{2}\int\frac{\sin\ x}{\cos\ x}dx=-\frac{1}{2}\ln|\cos\ x|$$

$$\Rightarrow \mu=\frac{1}{\sqrt{\cos x}}$$

同乘積分因子後可得

$$\frac{-xy\sin x+2y\cos x}{\sqrt{\cos x}}dx+2\frac{x\cos x}{\sqrt{\cos x}}dy=0$$

由於此微正合 ODE

$\therefore$ 可知 $\overline{N}=\dfrac{\partial\phi}{\partial y}=2x\sqrt{\cos x} \quad \Rightarrow \phi=2xy\sqrt{\cos x}+g(x)$ 代入

$$\overline{M}=\frac{\partial\phi}{\partial x}=2y\sqrt{\cos x}+xy\cdot\frac{-\sin x}{\sqrt{\cos x}}+g'(x)$$

比較後可得 $g'(x)=0 \quad \Rightarrow \phi(x,y)=2xy\sqrt{\cos x}=C$

3. 已知微分方程式為 $y^2-(1-2xy)y'=0$

(1) 此微分方程式為線性或非線性? (2%) 並以一階線性法求解。(7%)
(若為線性，直接求解；若非線性，則轉換成線性，再求解)

(2) 此微分方程式為正合(exact)或非正合? (2%) 並以正合法求解。(7%)
(若正合，直接求解；若非正合，先求出積分因子，再求解)

(1) $y^2-(1-2xy)y'=0 \quad \Rightarrow \dfrac{dy}{dx}=\dfrac{y^2}{1-2xy}$

$$\Rightarrow \frac{dx}{dy}=\frac{1-2xy}{y^2}$$

$$\Rightarrow \frac{dx}{dy}+\frac{2}{y}x=\frac{1}{y^2}$$

若以 x 為自變數，y 為應變數，則此為一階非線性微分方程式
若以 y 為自變數，x 為應變數，則此為一階線性微分方程式

積分因子 $\mu = e^{\int p(y)dy} = e^{\int \frac{2}{y}dy} = e^{2\ln|y|} = y^2$

同乘積分因子後可得

$$y^2 \frac{dx}{dy} + 2yx = 1 \quad \Rightarrow \frac{d}{dy}(y^2 x) = 1$$

$$\Rightarrow y^2 x = y + C$$

(2) $y^2 - (1-2xy)y' = 0 \quad \Rightarrow y^2 dx - (1-2xy)dy = 0$

令 $M = y^2 \quad \Rightarrow \frac{\partial M}{\partial y} = 2y$

$N = -(1-2xy) \quad \Rightarrow \frac{\partial N}{\partial x} = 2y$

由判斷式 $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ 可知，此為正合 ODE

$\therefore$ 可知 $M = \frac{\partial \phi}{\partial x} = y^2 \quad \Rightarrow \phi = xy^2 + f(y)$

$N = \frac{\partial \phi}{\partial y} = -(1-2xy) \quad \Rightarrow \phi = -y + xy^2 + g(x)$

比較後可得 $\phi(x,y) = -y + xy^2 = C$

4. 已知微分方程式為 $x^2 \frac{dy}{dx} + xy + x^2 y^2 = 1$

(1) 此為何種類型之微分方程式? (Clairaut、Bernoulli 或是 Riccati) (2%)

(2) 此為線性或非線性? (2%)

(3) 試求此微分方程式之解 $y(x) = ?$ **(7%)**

(1) $x^2 \frac{dy}{dx} + xy + x^2 y^2 = 1 \quad \Rightarrow \frac{dy}{dx} = -\frac{1}{x}y - y^2 + \frac{1}{x^2}$

此為 Riccati ODE

(2) 非線性

(3) 由觀察得一解 $S = -\frac{1}{x}$

令 $y = S + \frac{1}{V} = -\frac{1}{x} + \frac{1}{V} \quad \Rightarrow \frac{dy}{dx} = \frac{1}{x^2} - \frac{V'}{V^2}$ 代回 ODE 可得

$$\frac{1}{x^2} - \frac{V'}{V^2} = -\frac{1}{x}\left(-\frac{1}{x} + \frac{1}{V}\right) - \left(-\frac{1}{x} + \frac{1}{V}\right)^2 + \frac{1}{x^2}$$

$$\Rightarrow -\frac{V'}{V^2} = -\frac{1}{x}\frac{1}{V} + \frac{2}{x}\frac{1}{V} - \frac{1}{V^2}$$

$$\Rightarrow V' + \frac{1}{x}V = 1$$ ⟶ 此為一階線性 ODE

可知其積分因子為 $\mu = e^{\int p(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

同乘積分因子後可得 $xV' + V = x \quad \Rightarrow \frac{d}{dx}(xV) = x$

$$\Rightarrow xV = \frac{1}{2}x^2 + C_1$$

$$\Rightarrow V = \frac{1}{2}x + C_1\frac{1}{x}$$

$$\therefore\ y = S + \frac{1}{V} = -\frac{1}{x} + \frac{1}{\frac{1}{2}x + C_1\frac{1}{x}} = -\frac{1}{x} + \frac{2x}{x^2 + C}$$

5. 已知微分方程式為 $y = xy' + (y')^2$

(1) 此為何種類型之微分方程式? (Clairaut、Bernoulli 或是 Riccati) (2%)

(2) 此為線性或非線性? (2%)

(3) 試求此微分方程式之解 $y(x) = ?$ **(8%)**

(1) 此為 Clairaut ODE

(2) 屬於非線性 ODE

(3) $y = xy' + (y')^2$

令 $p = y' \quad \Rightarrow y = xp + p^2$

將兩邊對 x 微分可得 $y' = p + xp' + 2pp'$

$$\Rightarrow p = p + xp' + 2pp'$$

$$\Rightarrow (x + 2p)p' = 0$$

由 $p' = 0 \quad \Rightarrow y' = p = c$ 代回 $y = xy' + y'^2$

可得 $y = xc + c^2$（通解）

由 $x + 2p = 0 \quad \Rightarrow y' = p = -\frac{x}{2}$ 代回 $y = xy' + y'^2$

可得 $y = -\frac{x^2}{4}$（奇解）

6. 試解下列各微分方程

(1) $(x+1)y'+y=\ln x$，$y(1)=-1$ **(7%)** **(110 成大土木)**

(2) $(6x^6-x^2y)+(-x^3+x^2y^2)y'=0$，$y(0)=3$ **(7%)** **(110 台大土木)**

(3) $2(y')^2-(2y^2+x)y'+xy^2=0$ **(7%)**

(1) $(x+1)y'+y=\ln x \quad \Rightarrow \frac{d}{dx}[(x+1)y]=\ln x$

$$\Rightarrow (x+1)y=\int \ln x\,dx=-x+x\ln x+C$$

$$\Rightarrow y=\frac{1}{x+1}(-x+x\ln x+C)$$

又 $y(1)=-1 \quad \Rightarrow -1=\frac{1}{2}(-1+C) \quad \Rightarrow C=-1$

$\therefore\ y(x)=\frac{1}{x+1}(x\ln x-x-1)$

(2) $(6x^6-x^2y)+(-x^3+x^2y^2)y'=0$

$\Rightarrow (6x^6dx-x^2y\,dx)+(-x^3dy+x^2y^2dy)=0$

$\Rightarrow 6x^4dx-y\,dx-xdy+y^2dy=0$

$\Rightarrow 6x^4dx-d(xy)+y^2dy=0$

$\Rightarrow \int 6x^4dx-\int d(xy)+\int y^2dy=0$

$\Rightarrow \frac{6}{5}x^5-xy+\frac{1}{3}y^3=C$

又 $y(0)=3 \quad \Rightarrow C=9$

$\therefore\ \frac{6}{5}x^5-xy+\frac{1}{3}y^3=9$

(3) $2(y')^2-(2y^2+x)y'+xy^2=0$

$\Rightarrow (2y'-x)(y'-y^2)=0$

$\Rightarrow y'=\frac{x}{2}$ or $y'=y^2$

當 $y'=\frac{x}{2} \quad \Rightarrow y=\frac{x^2}{4}+C_1$

當 $y'=y^2 \quad \Rightarrow y^{-2}dy=dx \quad \Rightarrow -y^{-1}=x+C_2 \quad \Rightarrow y=-\frac{1}{x+C_2}$

$\therefore$ ODE 解為 $(y-\frac{x^2}{4}-C_1)\,(y+\frac{1}{x+C_2})=0$

7. 試以皮卡德法(Picard method)求解 $y' = 2y^2$，$y(0) = 1$ **(取至** y_2**)** **(8%)**

$$\frac{dy}{dx} = f(x,y) \quad \Rightarrow dy = f(x,y)\,dx$$

$$\Rightarrow \int_{y_0}^{y} dy = \int_{x_0}^{x} f(t, y(t))\,dt$$

$$\Rightarrow y = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt$$

$$y_0 = y(0) = 1$$

$$y_1 = y_0 + \int_{x_0}^{x} f(t, y_0(t))\,dt = 1 + \int_0^x 2\,dt = 1 + 2x$$

$$y_2 = y_0 + \int_{x_0}^{x} f(t, y_1(t))\,dt = 1 + \int_0^x 2(1+2t)^2\,dt = 1 + 2x + 4x^2 + \frac{8}{3}x^3$$

$$y_3 = y_0 + \int_{x_0}^{x} f(t, y_2(t))\,dt = 1 + \int_0^x 2(1 + 2t + 4t^2 + \frac{8}{3}t^3)^2\,dt = 1 + 2x + 4x^2 + 8x^3$$

$$+ \frac{32}{3}x^4 + \frac{32}{3}x^5 + \frac{64}{9}x^6 + \frac{128}{63}x^7$$

$$y = \lim_{n\to\infty} y_n$$

另解： $y' = 2y^2 \quad \Rightarrow y^{-2}dy = 2dx \quad \Rightarrow -y^{-1} = 2x + C \quad \Rightarrow y = -\dfrac{1}{2x+C}$

又 $y(0) = 1 \quad \Rightarrow 1 = -\dfrac{1}{C} \quad \Rightarrow C = -1$

$\therefore \quad y = \dfrac{1}{1-2x} = 1 + 2x + 4x^2 + 8x^3 + 16x^4 + \cdots$