國立臺灣海洋大學河海工程學系 2020 工程數學(一) 第二次隨堂測驗參考解答

系級：＿＿＿＿＿＿　　學號：＿＿＿＿＿＿　　姓名：＿＿＿＿＿＿

1. 試求下述 Bernoulli ODE: $3(1+x^2)y' = 2xy(y^3-1)$

$$3(1+x^2)y' = 2xy(y^3-1) \Rightarrow y' + \frac{2x}{3(1+x^2)}y = \frac{2x}{3(1+x^2)}y^4$$

$$\Rightarrow y^{-4}y' + \frac{2x}{3(1+x^2)}y^{-3} = \frac{2x}{3(1+x^2)}$$

此為 Bernoulli ODE

令 $u = y^{-3} \quad \Rightarrow u' = -3y^{-4}y'$ 代回 ODE 可得

$$-\frac{1}{3}u' + \frac{2x}{3(1+x^2)}u = \frac{2x}{3(1+x^2)} \quad \Rightarrow u' - \frac{2x}{1+x^2}u = -\frac{2x}{1+x^2}$$

此為一階線性 ODE，可得積分因子為 $\mu = e^{\int p(x)dx} = e^{-\int \frac{2x}{1+x^2}dx} = e^{-\ln(1+x^2)} = \frac{1}{1+x^2}$

同乘積分因子後可得 $\frac{1}{1+x^2}u' - \frac{2x}{(1+x^2)^2}u = -\frac{2x}{(1+x^2)^2}$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{1+x^2}u\right) = -\frac{2x}{(1+x^2)^2}$$

$$\Rightarrow \frac{1}{1+x^2}u = -\int \frac{2x}{(1+x^2)^2}\,dx = \frac{1}{1+x^2} + C$$

$$\Rightarrow u = 1 + C(1+x^2)$$

$$\Rightarrow \frac{1}{y^3} = 1 + C(1+x^2)$$

$$\Rightarrow e^{4x}(y^{-4} + x - \frac{1}{4}) = C$$

2. 試求下述 Riccati ODE: $y' = y^2 - 2xy + x^2 + 1$

$y' = y^2 - 2xy + x^2 + 1 \longrightarrow$ 此為 Riccati ODE

由觀察得一解為 $S = x$

令 $y = S + \frac{1}{V} = x + \frac{1}{V} \quad \Rightarrow y' = 1 - \frac{V'}{V^2}$ 代回 ODE 可得

$$1 - \frac{V'}{V^2} = (x + \frac{1}{V})^2 - 2x(x + \frac{1}{V}) + x^2 + 1 \Rightarrow 1 - \frac{V'}{V^2} = x^2 + \frac{2x}{V} + \frac{1}{V^2} - 2x^2 - \frac{2x}{V} + x^2 + 1$$

$$\Rightarrow V' = -1 \quad \Rightarrow V = -x + C$$

$\therefore \quad y = S + V = x + \frac{1}{C - x}$