國立臺灣海洋大學河海工程學系 2020 工程數學(一) 第一次隨堂測驗參考解答

系級：＿＿＿＿＿＿＿＿　　學號：＿＿＿＿＿＿＿　　姓名：＿＿＿＿＿＿

給一微分方程式 $xy^2y' = x^3 + y^3$

(1) 試以變數變換法求解 (齊次型)

(2) 試以正合法求解

(1) $xy^2y' = x^3 + y^3 \Rightarrow y' = \frac{x^2}{y^2} + \frac{y}{x}$

令 $u = \frac{y}{x} \Rightarrow y = ux \quad \Rightarrow \frac{dy}{dx} = \frac{du}{dx}\cdot x + u$

$$\therefore \frac{du}{dx}\cdot x + u = \frac{1}{u^2} + u \quad \Rightarrow \frac{du}{dx}\cdot x = \frac{1}{u^2} \quad \Rightarrow \int u^2 du = \int \frac{1}{x} dx$$

$$\Rightarrow \frac{1}{3}u^3 = \ln|x| + C$$

$$\Rightarrow \frac{1}{3}\frac{y^3}{x^3} = \ln|x| + C$$

(2) $xy^2y' = x^3 + y^3 \Rightarrow (x^3 + y^3)dx - xy^2 dy = 0$

令 $M = x^3 + y^3 \quad \Rightarrow \frac{\partial M}{\partial y} = 3y^2$

$N = -xy^2 \quad \Rightarrow \frac{\partial N}{\partial x} = -y^2$

由判斷式 $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ 此為非正合方程式

由 $\frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) = -\frac{4}{x}$ 可知積分因子 $\mu = \mu(x)$

故可得 $\mu = e^{-\int \frac{4}{x}dx} = \frac{1}{x^4}$

同乘積分因子後 $(\frac{1}{x} + \frac{y^3}{x^4})dx - \frac{y^2}{x^3}dy = 0$ 此為正合微分方程

故可知 $\frac{\partial \phi}{\partial x} = \frac{1}{x} + \frac{y^3}{x^4} \quad \Rightarrow \phi = \ln|x| - \frac{1}{3}\frac{y^3}{x^3} + f(y)$

$\frac{\partial \phi}{\partial y} = -\frac{y^2}{x^3} \quad \Rightarrow \phi = -\frac{1}{3}\frac{y^3}{x^3} + g(x)$

比較上兩式後可得 $\phi(x,y) = \ln|x| - \frac{1}{3}\frac{y^3}{x^3} = C$