

系級：_____ 學號：_____ 姓名：_____

1. $\mathbf{A} = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$, 請使用 Gram-Schmidt 法針對行向量空間求出一組單位正交基底向量。

2. $\mathbf{A} = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 3 & 3 \\ 2 & 1 & -2 \\ 3 & 3 & 3 \\ 2 & -2 & 1 \\ 3 & -3 & 3 \end{bmatrix}$, 試求各矩陣之特徵值與特徵向量。

3. $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 7 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -6 & 6 & 0 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 2 & 4 \end{bmatrix}$, 試求各矩陣之特徵值與特徵向量。

參考解答：

1. $u_1 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)^T$

$u_2 = \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)^T$

$u_3 = \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)^T$

2. $|\mathbf{A} - \lambda\mathbf{I}| = 0 \Rightarrow \begin{vmatrix} 3-\lambda & 0 & -1 \\ 0 & 1-\lambda & 0 \\ -1 & 0 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda-1)(\lambda-2)(\lambda-4) = 0 \Rightarrow \lambda = 1, 2, 4$

當 $\lambda_1 = 1$ 時，可得 $\mathbf{x}^1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

當 $\lambda_2 = 2$ 時，可得 $\mathbf{x}^2 = \begin{Bmatrix} 1 \\ \sqrt{2} \\ 0 \\ 1 \\ \sqrt{2} \end{Bmatrix}$

當 $\lambda_3 = 4$ 時，可得 $\mathbf{x}^3 = \begin{Bmatrix} 1 \\ \sqrt{2} \\ 0 \\ 1 \\ -\sqrt{2} \end{Bmatrix}$

$$|\mathbf{B} - \lambda\mathbf{I}| = 0 \Rightarrow \begin{vmatrix} \frac{1}{3} - \lambda & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} - \lambda & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} - \lambda \end{vmatrix} = 0 \Rightarrow (\lambda - 1)^2(\lambda + 1) = 0 \Rightarrow \lambda = 1, 1, -1$$

當 $\lambda_1 = -1$ 時，可得 $\mathbf{x}^1 = \begin{Bmatrix} -1 \\ 1 \\ 1 \end{Bmatrix}$

當 $\lambda_2 = \lambda_3 = 1$ 時，可得 $\mathbf{x}^2 = \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}$ ， $\mathbf{x}^3 = \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix}$

3. $|\mathbf{A} - \lambda\mathbf{I}| = 0 \Rightarrow \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 7 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ -6 & 6 & 0 & 1 - \lambda \end{vmatrix} = 0$
 $\Rightarrow (\lambda - 1)(\lambda + 1)(\lambda + 2)(\lambda - 3) = 0 \Rightarrow \lambda = 1, -1, -2, 3$

當 $\lambda_1 = 1$ 時，可得 $\mathbf{x}^1 = \begin{Bmatrix} 1 \\ 1 \\ -6 \\ -6 \end{Bmatrix}$

當 $\lambda_2 = -1$ 時，可得 $\mathbf{x}^2 = \begin{Bmatrix} 1 \\ -1 \\ -6 \\ 6 \end{Bmatrix}$

$$\text{當 } \lambda_3 = -2 \text{ 時，可得 } \mathbf{x}^3 = \begin{Bmatrix} 1 \\ -2 \\ -3 \\ 6 \end{Bmatrix}$$

$$\text{當 } \lambda_4 = 3 \text{ 時，可得 } \mathbf{x}^1 = \begin{Bmatrix} 1 \\ 3 \\ 2 \\ 6 \end{Bmatrix}$$

$$|\mathbf{B} - \lambda\mathbf{I}| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & 3-\lambda & 6 \\ 0 & 0 & 2 & 4-\lambda \end{vmatrix} = 0$$
$$\Rightarrow \lambda(\lambda-1)^2(\lambda-7) = 0 \Rightarrow \lambda = 0, 1, 1, 7$$

$$\text{當 } \lambda_1 = 0 \text{ 時，可得 } \mathbf{x}^1 = \begin{Bmatrix} 0 \\ 0 \\ 2 \\ -1 \end{Bmatrix}$$

$$\text{當 } \lambda_2 = \lambda_3 = 1 \text{ 時，可得 } \mathbf{x}^2 = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \quad \mathbf{x}^3 = \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix}$$

$$\text{當 } \lambda_4 = 7 \text{ 時，可得 } \mathbf{x}^1 = \begin{Bmatrix} 0 \\ 0 \\ 3 \\ 2 \end{Bmatrix}$$