國立臺灣海洋大學河海工程學系 2020 工程數學(一) 2A 班第八次作業參考解答

系級：________　　學號：________　　姓名：________

1. 試解:

(1) $(2x+1)^2 y''-(12x+6)y'+16y=2$

(2) $x(x-1)y''+xy'-y=0$

(3) $x(x+1)y''+(4x+1)y'+2y=2x+1$

2. (1) 已知 $y=e^{2x}$ 為方程式 $(x+2)y''-(2x+5)y'+2y=0$ 之解，試求此方程式之通解。

(2) 已知微分方程 $2xy''+(1-4x)y'+(2x-1)y=e^x$，試求此方程式之通解。

3. 試解:

(1) $xy''-y'-(3+x)x^2e^x=0$ 且 $y(0)=0$、$y(1)=2e$

(2) $yy''+3(y')^2=0$

(3) $xyy''+x(y')^2+2yy'=0$

參考解答:

1. (1) $(2x+1)^2 y''-(12x+6)y'+16y=2$

令 $u=2x+1 \Rightarrow \dfrac{du}{dx}=2$

$$y'=\frac{dy(x)}{dx}=\frac{du}{dx}\cdot\frac{dY(u)}{du}=2Y'(u)$$

$$y''=\frac{dy'(x)}{dx}=\frac{du}{dx}\cdot\frac{d(2Y'(u))}{du}=4Y''(u)$$ 代回 ODE 可得

$4u^2Y''-12uY'+16Y=2 \longrightarrow$ 此為 Euler-Cauchy ODE

令 $Y=u^m$ 代入 ODE 可得特徵方程式

$$4m(m-1)-12m+16=0 \Rightarrow m^2-4m+4=0$$
$$\Rightarrow (m-2)^2=0$$
$$\Rightarrow m=2,2$$

$\therefore\ Y_h=C_1u^2+C_2u^2\cdot\ln u$

令 $Y_p=v_1Y_1+v_2Y_2$

可知 $W=\begin{vmatrix}Y_1 & Y_2\\ Y_1' & Y_2'\end{vmatrix}=\begin{vmatrix}u^2 & u^2\cdot\ln u\\ 2u & 2u\ln u+u\end{vmatrix}=u^3$

$$v_1' = \frac{\begin{vmatrix} 0 & Y_2 \\ \frac{f(x)}{4u^2} & Y_2' \end{vmatrix}}{W} = \frac{\begin{vmatrix} 0 & u^2 \cdot \ln u \\ \frac{1}{2u^2} & 2u\ln u + u \end{vmatrix}}{u^3} = -\frac{1}{2u^3}\ln u \quad \Rightarrow v_1 = \frac{1}{8u^2} + \frac{1}{4u^2}\ln u$$

$$v_1' = \frac{\begin{vmatrix} Y_1 & 0 \\ Y_1' & \frac{f(x)}{4u^2} \end{vmatrix}}{W} = \frac{\begin{vmatrix} u^2 & 0 \\ 2u & \frac{1}{2u^2} \end{vmatrix}}{u^3} = \frac{1}{2u^3} \quad \Rightarrow v_2 = -\frac{1}{4u^2}$$

$$\therefore \ Y_p = v_1 Y_1 + v_2 Y_2 = (\frac{1}{8u^2} + \frac{1}{4u^2}\ln u) \cdot u^2 + (-\frac{1}{4u^2}) \cdot u^2 \cdot \ln u = \frac{1}{8}$$

故可得通解為 $Y(u) = Y_h + Y_p = C_1 u^2 + C_2 u^2 \cdot \ln u + \frac{1}{8}$

$$= C_1(2x+1)^2 + C_2(2x+1)^2 \cdot \ln|2x+1| + \frac{1}{8}$$

(2) $x(x-1)y'' + xy' - y = 0$

令 $a_2 = x(x-1)$，$a_1 = x$，$a_0 = -1$

由 $a_2'' - a_1' + a_0 = 2 - 1 - 1 = 0$ 可知此為正合

$$\therefore \ x(x-1)y'' + xy' - y = \frac{d}{dx}(b_1 y' + b_0 y) = b_1 y'' + (b_1' + b_0)y' + b_0' y$$

比較係數可知 $b_1 = x(x-1)$，$b_0 = -x+1$

$$\therefore \ \frac{d}{dx}[x(x-1)y' - (x-1)y] = 0 \quad \Rightarrow x(x-1)y' - (x-1)y = C_1$$

$$\Rightarrow y' - \frac{1}{x}y = \frac{C_1}{x(x-1)}$$ 此為一階線性 ODE

可知積分因子 $\mu = e^{-\int \frac{1}{x}dx} = e^{-\ln|x|} = \frac{1}{x}$

同乘積分因子可得 $\frac{1}{x}y' - \frac{1}{x^2}y = \frac{C_1}{x^2(x-1)}$

$$\Rightarrow \frac{d}{dx}(\frac{1}{x}y) = \frac{C_1}{x^2(x-1)}$$

$$\Rightarrow \frac{1}{x}y = C_1(\frac{1}{x} + \ln|x-1| - \ln|x|) + C_2$$

$$\Rightarrow y = C_1(1 + x\ln|x-1| - x\ln|x|) + C_2 x$$

(3) $x(x+1)y'' + (4x+1)y' + 2y = 2x+1$

令 $a_2 = x(x-1)$，$a_1 = 4x+1$，$a_0 = 2$

由 $a_2'' - a_1' + a_0 = 2 - 4 + 2 = 0$ 可知此為正合

$$\therefore \quad x(x+1)y''+(4x+1)y'+2y=\frac{d}{dx}(b_1y'+b_0y)=b_1y''+(b_1'+b_0)y'+b_0'y$$

比較係數可知 $b_1=x(x+1)$，$b_0=2x$

$$\therefore \quad \frac{d}{dx}[x(x+1)y'+2xy]=2x+1$$

$$\Rightarrow x(x+1)y'+2xy=x^2+x+C_1$$

$$\Rightarrow y'+\frac{2}{x+1}y=1+\frac{C_1}{x(x+1)}$$ 此為一階線性 ODE

可知積分因子 $\mu=e^{\int\frac{2}{x+1}dx}=e^{2\ln|x+1|}=(x+1)^2$

同乘積分因子可得 $(x+1)^2y'+2(x+1)y=(x+1)^2+C_1\dfrac{x+1}{x}$

$$\Rightarrow \frac{d}{dx}[(x+1)^2y]=(x+1)^2+C_1\frac{x+1}{x}$$

$$\Rightarrow (x+1)^2y=\frac{1}{3}(x+1)^3+C_1(x+\ln|x|)+C_2$$

$$\Rightarrow y=\frac{1}{3}(x+1)+\frac{1}{(x+1)^2}[C_1(x+\ln|x|)+C_2]$$

2. (1) $(x+2)y''-(2x+5)y'+2y=0 \quad \Rightarrow y''-\dfrac{2x+5}{x+2}y'+\dfrac{2}{x+2}y=0$

$$W(y_1,y_2)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=y_1y_2'-y_1'y_2$$

且滿足 $W'-\dfrac{2x+5}{x+2}W=0 \quad \Rightarrow \dfrac{W'}{W}=\dfrac{2x+5}{x+2}$

$$\Rightarrow \ln W=\int(2+\frac{1}{x+2})dx=2x+\ln|x+2|+\overline{C}_1$$

$$\Rightarrow W=e^{2x+\ln|x+2|+\overline{C}_1}=C_1(x+2)e^{2x}$$

$$\therefore \quad y_1y_2'-y_1'y_2=C_1(x+2)e^{2x}$$

$$\Rightarrow e^{2x}\cdot y_2'-2e^{2x}y_2=C_1(x+2)e^{2x}$$

$$\Rightarrow y_2'-2y_2=C_1(x+2) \longrightarrow$$ 為一階線性 ODE

積分因子為 $\mu=e^{-\int 2dx}=e^{-2x}$

同乘積分因子後可得 $e^{-2x}y_2'-2e^{-2x}y_2=C_1(x+2)e^{-2x}$

$$\Rightarrow \frac{d}{dx}(e^{-2x}y_2)=C_1(x+2)e^{-2x}$$

$$\Rightarrow e^{-2x}y_2=C_1(-\frac{x}{2}-\frac{5}{4})e^{-2x}+C_2$$

$$\Rightarrow e^{-2x}y_2 = C_1(-\frac{x}{2}-\frac{5}{4})e^{-2x}+C_2$$

$\therefore$ 另一解為 $\frac{x}{2}+\frac{5}{4}$

(2) $2xy''+(1-4x)y'+(2x-1)y=e^x$

先求補解滿足 $2xy''+(1-4x)y'+(2x-1)y=0$

由觀察可得一補解 e^x

$$W(y_1,y_2)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2$$

且滿足 $W'+\frac{1-4x}{2x}W=0 \quad \Rightarrow \frac{W'}{W}=-\frac{1-4x}{2x}$

$$\Rightarrow \ln W = \int(2-\frac{1}{2x})dx = 2x-\frac{1}{2}\ln|2x|+\overline{C}_1$$

$$\Rightarrow W = e^{2x-\frac{1}{2}\ln|2x|+\overline{C}_1} = C_1\frac{e^{2x}}{\sqrt{2x}}$$

$$\therefore \quad y_1y_2' - y_1'y_2 = C_1\frac{e^{2x}}{\sqrt{2x}}$$

$$\Rightarrow e^x\cdot y_2' - e^x y_2 = C_1\frac{e^{2x}}{\sqrt{2x}}$$

$$\Rightarrow y_2' - y_2 = C_1\frac{e^{x}}{\sqrt{2x}} \longrightarrow \text{為一階線性 ODE}$$

積分因子為 $\mu = e^{-\int dx} = e^{-x}$

同乘積分因子後可得 $e^{-x}y_2' - e^{-x}y_2 = \overline{C}_1\frac{1}{\sqrt{2x}}$

$$\Rightarrow \frac{d}{dx}(e^{-x}y_2) = \overline{C}_1\frac{1}{\sqrt{2x}}$$

$$\Rightarrow e^{-x}y_2 = \overline{C}_1\sqrt{2x}+C_2$$

$$\Rightarrow y_2 = C_1e^x\sqrt{x}+C_2e^x$$

$\therefore$ 另一解為 $e^x\sqrt{x}$

由參數變異法可令 $y_p = u_1y_1+u_2y_2$

可知 $W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^x\sqrt{x} \\ e^x & e^x(\sqrt{x}+\frac{1}{2\sqrt{x}}) \end{vmatrix} = \frac{1}{2\sqrt{x}}e^{2x}$

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ \frac{f(x)}{2x} & y_2' \end{vmatrix}}{W} = \frac{\begin{vmatrix} 0 & e^x\sqrt{x} \\ \frac{e^x}{2x} & e^x(\sqrt{x}+\frac{1}{2\sqrt{x}}) \end{vmatrix}}{\frac{1}{2\sqrt{x}}e^{2x}} = -1 \quad \Rightarrow u_1 = -x$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & \frac{f(x)}{2x} \end{vmatrix}}{W} = \frac{\begin{vmatrix} e^x & 0 \\ e^x & \frac{e^x}{2x} \end{vmatrix}}{\frac{1}{2\sqrt{x}}e^{2x}} = \frac{1}{\sqrt{x}} \quad \Rightarrow u_2 = 2\sqrt{x}$$

$\therefore \quad y_p = u_1 y_1 + u_2 y_2 = (-x)\cdot e^x + 2\sqrt{x}\cdot e^x\sqrt{x} = xe^x$

故可得通解為 $y = y_h + y_p = C_1 e^x + C_2 e^x\sqrt{x} + xe^x = e^x(C_1 + C_2\sqrt{x} + x)$

3. (1) $xy'' - y' - (3+x)x^2 e^x = 0$ 且 $y(0)=0$ 、 $y(1)=2e$

令 $u = y' \quad \Rightarrow u' = y''$

原式 $\Rightarrow \quad xu' - u - (3+x)x^2 e^x = 0$

$\Rightarrow \quad u' - \frac{1}{x}u = (3+x)xe^x$ ⟶ 為一階線性 ODE

積分因子為 $\mu = e^{-\int \frac{1}{x}dx} = \frac{1}{x}$

同乘積分因子後可得 $\frac{1}{x}u' - \frac{1}{x^2}u = (3+x)e^x$

$$\Rightarrow \frac{d}{dx}(\frac{1}{x}u) = (3+x)e^x$$

$$\Rightarrow \frac{1}{x}u = (2+x)e^x + \overline{C}_1$$

$$\Rightarrow u = (2+x)xe^x + \overline{C}_1 x$$

$$\Rightarrow y' = (2+x)xe^x + C_1 x^2$$

$$\Rightarrow y = x^2 e^x + C_1 x^2 + C_2$$

又 $y(0)=0 \quad \Rightarrow C_2 = 0$

$y(1) = 2e \quad \Rightarrow C_1 = e$

$\therefore \quad y = x^2(e^x + e)$

(2) $yy'' + 3(y')^2 = 0$

令 $u = y' \quad \Rightarrow y'' = \frac{dy'}{dx} = \frac{dy}{dx}\frac{dy'}{dy} = u\frac{du}{dy}$

原式 $\Rightarrow \quad yu\dfrac{du}{dy}+3u^2=0 \quad \Rightarrow \quad u(y\dfrac{du}{dy}+3u)=0$

$$\Rightarrow \quad u=0 \quad \text{or} \quad y\frac{du}{dy}+3u$$

當 $u=0 \quad \Rightarrow y'=0 \quad \Rightarrow y=C$

當 $y\dfrac{du}{dy}+3u=0 \quad \Rightarrow -\dfrac{3}{y}dy=\dfrac{1}{u}du$

$$\Rightarrow -\int\frac{3}{y}dy=\int\frac{1}{u}du$$

$$\Rightarrow -3\ln|y|=\ln|u|+\overline{C}_1$$

$$\Rightarrow \frac{C_1}{y^3}=u$$

$$\Rightarrow \frac{C_1}{y^3}=y'$$

$$\Rightarrow \int y^3dy=\int C_1dx$$

$$\Rightarrow \frac{1}{4}y^4=C_1x+C_2$$

$\therefore$ 此 ODE 之解為 $\dfrac{1}{4}y^4=C_1x+C_2$ 或 $y=C$

(3) $xyy''+x(y')^2+2yy'=0$

令 $y=e^z \quad \Rightarrow y'=\dfrac{de^z}{dx}=\dfrac{dz}{dx}\dfrac{de^z}{dz}=e^zz'$

$$\Rightarrow y''=\frac{dy'}{dx}=e^z(z')^2+e^zz''$$

原式 $\Rightarrow \quad xe^z[e^z(z')^2+e^zz'']+x(e^zz')^2+2e^ze^zz'=0$

$\Rightarrow \quad xz''+2x(z')^2+2z'=0$

令 $u=z' \quad \Rightarrow u'=z''$

代入上式可得 $u'+\dfrac{2}{x}u=-2u^2$ ⟶ 此為 Bernoulli ODE

$$\Rightarrow u^{-2}u'+\frac{2}{x}u^{-1}=-2$$

令 $v=u^{-1} \quad \Rightarrow v'=-u^{-2}u'$

代入上式可得 $-v'+\dfrac{2}{x}v=-2 \quad \Rightarrow v'-\dfrac{2}{x}v=2$ ⟶ 此為一階線性 ODE

積分因子為 $\mu=e^{-\int\frac{2}{x}dx}=\dfrac{1}{x^2}$

同乘積分因子後可得 $\frac{1}{x^2}v' - \frac{2}{x^3}v = \frac{2}{x^2}$

$$\Rightarrow \frac{d}{dx}(\frac{1}{x^2}v) = \frac{2}{x^2}$$

$$\Rightarrow \frac{1}{x^2}v = -\frac{2}{x} + C_1$$

$$\Rightarrow v = -2x + C_1 x^2$$

$$\Rightarrow u = \frac{1}{-2x + C_1 x^2}$$

$$\Rightarrow z = \int \frac{1}{-2x + C_1 x^2} dx = -\frac{1}{2}\ln|x| + \frac{1}{2}\ln|C_1 x - 2| \mp C_2$$

$$\Rightarrow \ln|y| = -\frac{1}{2}\ln|x| + \frac{1}{2}\ln|C_1 x - 2| + \overline{C}_2$$

$$\Rightarrow y = C_2 \frac{\sqrt{C_1 x - 2}}{\sqrt{x}}$$