

系級：\_\_\_\_\_ 學號：\_\_\_\_\_ 姓名：\_\_\_\_\_

試解：

1.  $x^3 y''' + x^2 y'' - 2xy' + 2y = x^{-2}$
2.  $x^2 y'' - 3xy' + 4y = 3x^2$
3.  $xy'' - y' + 2x^{-1}y = 2 \cos(\ln x)$
4.  $x^2 y'' - 2xy' + 2y = x \ln x$
5.  $x^2 y'' - 4xy' + 6y = -7x^4 \sin x$
6.  $xy'' - y' = 2x^2 e^x$
7.  $x^2 y'' + xy' + y = \sec(\ln x)$

參考解答：

1.  $x^3 y''' + x^2 y'' - 2xy' + 2y = x^{-2}$

令  $y = x^m$  代入 ODE 可得特徵方程式

$$\begin{aligned} m(m-1)(m-2) + m(m-1) - 2m + 2 &= 0 \Rightarrow m^3 - 2m^2 - m + 2 = 0 \\ &\Rightarrow (m-1)(m+1)(m-2) = 0 \\ &\Rightarrow m = 1, -1, 2 \end{aligned}$$

$$\therefore y_h = C_1 x + C_2 x^{-1} + C_3 x^2$$

令  $y_p = u_1 y_1 + u_2 y_2 + u_3 y_3$

$$\text{可知 } W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} x & x^{-1} & x^2 \\ 1 & -x^{-2} & 2x \\ 0 & 2x^{-3} & 2 \end{vmatrix} = -6x^{-1}$$

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ \frac{f(x)}{x^2} & y_2'' & y_3'' \end{vmatrix}}{W} = \frac{\begin{vmatrix} 0 & x^{-1} & x^2 \\ 0 & -x^{-2} & 2x \\ x^{-5} & 2x^{-3} & 2 \end{vmatrix}}{-6x^{-2}} = \frac{3x^{-5}}{-6x^{-1}} = -\frac{1}{2}x^{-4} \Rightarrow u_1 = \frac{1}{6}x^{-3}$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & \frac{f(x)}{x^2} & y_3'' \end{vmatrix}}{W} = \frac{\begin{vmatrix} x & 0 & x^2 \\ 1 & 0 & 2x \\ 0 & x^{-5} & 2 \end{vmatrix}}{-6x^{-1}} = \frac{-x^{-3}}{-6x^{-1}} = \frac{1}{6}x^{-2} \Rightarrow u_2 = -\frac{1}{6}x^{-1}$$

$$u_3' = \frac{\begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & \frac{f(x)}{x^2} \end{vmatrix}}{W} = \frac{\begin{vmatrix} x & x^{-1} & 0 \\ 1 & -x^{-2} & 0 \\ 0 & 2x^{-3} & x^{-5} \end{vmatrix}}{-6x^{-1}} = \frac{-2x^{-6}}{-6x^{-1}} = \frac{1}{3}x^{-5} \Rightarrow u_3 = -\frac{1}{12}x^{-4}$$

$$\therefore y_p = u_1 y_1 + u_2 y_2 + u_3 y_3 = \frac{1}{6}x^{-3} \cdot x - \frac{1}{6}x^{-1} \cdot x^{-1} - \frac{1}{12}x^{-4} \cdot x^2 = -\frac{1}{12}x^{-2}$$

$$\text{故可得通解為 } y = y_h + y_p = C_1 x + C_2 x^{-1} + C_3 x^2 - \frac{1}{12}x^{-2}$$

2.  $x^2 y'' - 3xy' + 4y = 3x^2$

令  $y = x^m$  代入 ODE 可得特徵方程式

$$m(m-1) - 3m + 4 = 0 \Rightarrow m^2 - 4m + 4 = 0$$

$$\Rightarrow (m-2)^2 = 0$$

$$\Rightarrow m = 2, 2$$

$$\therefore y_h = C_1 x^2 + C_2 x^2 \cdot \ln x$$

令  $y_p = u_1 y_1 + u_2 y_2$

$$\text{可知 } W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^2 \ln x \\ 2x & 2x \ln x + x \end{vmatrix} = x^3$$

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ \frac{f(x)}{x^2} & y_2' \end{vmatrix}}{W} = \frac{\begin{vmatrix} 0 & x^2 \ln x \\ 3 & 2x \ln x + x \end{vmatrix}}{x^3} = -\frac{3}{x} \ln x \Rightarrow u_1 = -\frac{3}{2}(\ln x)^2$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & \frac{f(x)}{x^2} \end{vmatrix}}{W} = \frac{\begin{vmatrix} x^2 & 0 \\ 2x & 3 \end{vmatrix}}{x^3} = \frac{3}{x} \Rightarrow u_2 = 3 \ln x$$

$$\therefore y_p = u_1 y_1 + u_2 y_2 = -\frac{3}{2}(\ln x)^2 \cdot x^2 + 3 \ln x \cdot x^2 \ln x = \frac{3}{2}x^2(\ln x)^2$$

$$\text{故可得通解為 } y = y_h + y_p = C_1 x^2 + C_2 x^2 \cdot \ln x + \frac{3}{2}x^2(\ln x)^2$$

3.  $xy'' - y' + 2x^{-1}y = 2 \cos(\ln x) \Rightarrow x^2 y'' - xy' + 2y = 2x \cos(\ln x)$

令  $y = x^m$  代入 ODE 可得特徵方程式

$$m(m-1) - m + 2 = 0 \Rightarrow m^2 - 2m + 2 = 0$$

$$\Rightarrow m = \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 2}}{2} = 1 \pm i$$

$$\therefore y_h = x[C_1 \cos(\ln x) + C_2 \sin(\ln x)]$$

$$\text{令 } y_p = u_1 y_1 + u_2 y_2$$

$$\text{可知 } W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x \cos(\ln x) & x \sin(\ln x) \\ \cos(\ln x) - \sin(\ln x) & \sin(\ln x) + \cos(\ln x) \end{vmatrix} = x$$

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{W} = \frac{\begin{vmatrix} 0 & x \sin(\ln x) \\ 2x^{-1} \cos(\ln x) & \sin(\ln x) + \cos(\ln x) \end{vmatrix}}{x} = -\frac{2}{x} \cos(\ln x) \cdot \sin(\ln x)$$

$$\Rightarrow u_1 = -2 \int \frac{1}{x} \cos(\ln x) \cdot \sin(\ln x) dx = -2 \int \cos(\ln x) \cdot \sin(\ln x) d(\ln x) = \frac{1}{2} \cos(2 \ln x)$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{W} = \frac{\begin{vmatrix} x \cos(\ln x) & 0 \\ \cos(\ln x) - \sin(\ln x) & 2x^{-1} \cos(\ln x) \end{vmatrix}}{x} = \frac{2}{x} \cos(\ln x) \cdot \cos(\ln x)$$

$$\Rightarrow u_2 = 2 \int \frac{1}{x} \cos(\ln x) \cdot \cos(\ln x) dx = 2 \int \cos(\ln x) \cdot \cos(\ln x) d(\ln x) = \ln x + \frac{1}{2} \sin(2 \ln x)$$

$$\begin{aligned} \therefore y_p &= u_1 y_1 + u_2 y_2 = \frac{1}{2} \cos(2 \ln x) \cdot x \cos(\ln x) + \left( \ln x + \frac{1}{2} \sin(2 \ln x) \right) \cdot x \sin(\ln x) \\ &= \frac{1}{2} x \cdot \cos(\ln x) + x \ln x \cdot \sin(\ln x) \end{aligned}$$

$$\text{故可得通解為 } y = y_h + y_p = x[C_1 \cos(\ln x) + C_2 \sin(\ln x)] + x \ln x \cdot \sin(\ln x)$$

$$4. \quad x^2 y'' - 2xy' + 2y = x \ln x$$

令  $y = x^m$  代入 ODE 可得特徵方程式

$$\begin{aligned} m(m-1) - 2m + 2 &= 0 \Rightarrow m^2 - 3m + 2 = 0 \\ &\Rightarrow (m-1)(m-2) = 0 \\ &\Rightarrow m = 1, 2 \end{aligned}$$

$$\therefore y_h = C_1 x + C_2 x^2$$

$$\text{令 } y_p = u_1 y_1 + u_2 y_2$$

$$\text{可知 } W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2$$

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{W} = \frac{\begin{vmatrix} 0 & x^2 \\ x^{-1} \ln x & 2x \end{vmatrix}}{x^2} = -\frac{1}{x} \ln x$$

$$\Rightarrow u_1 = -\int \frac{1}{x} \ln x \, dx = -\int \ln x \, d(\ln x) = -\frac{1}{2}(\ln x)^2$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & \frac{f(x)}{x^2} \end{vmatrix}}{W} = \frac{\begin{vmatrix} x & 0 \\ 1 & x^{-1} \ln x \end{vmatrix}}{x^2} = \frac{1}{x^2} \ln x \quad \Rightarrow u_2 = \int \frac{1}{x^2} \ln x \, dx = -\frac{1}{x} - \frac{1}{x} \ln x$$

$$\begin{aligned} \therefore y_p &= u_1 y_1 + u_2 y_2 = -\frac{1}{2}(\ln x)^2 \cdot x - \left(\frac{1}{x} + \frac{1}{x} \ln x\right) \cdot x^2 \\ &= -\frac{1}{2}x(\ln x)^2 - x \ln x - x \end{aligned}$$

故可得通解為  $y = y_h + y_p = C_1 x + C_2 x^2 - \frac{1}{2}x(\ln x)^2 - x \ln x$

5.  $x^2 y'' - 4xy' + 6y = -7x^4 \sin x$

令  $y = x^m$  代入 ODE 可得特徵方程式

$$\begin{aligned} m(m-1) - 4m + 6 &= 0 \quad \Rightarrow m^2 - 5m + 6 = 0 \\ &\Rightarrow (m-2)(m-3) = 0 \\ &\Rightarrow m = 2, 3 \end{aligned}$$

$$\therefore y_h = C_1 x^2 + C_2 x^3$$

令  $y_p = u_1 y_1 + u_2 y_2$

可知  $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = x^4$

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ \frac{f(x)}{x^2} & y_2' \end{vmatrix}}{W} = \frac{\begin{vmatrix} 0 & x^3 \\ -7x^2 \sin x & 3x^2 \end{vmatrix}}{x^4} = 7x \sin x$$

$$\Rightarrow u_1 = 7 \int x \sin x \, dx = -7x \cos x + 7 \sin x$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & \frac{f(x)}{x^2} \end{vmatrix}}{W} = \frac{\begin{vmatrix} x^2 & 0 \\ 2x & -7x^2 \sin x \end{vmatrix}}{x^4} = -7 \sin x \quad \Rightarrow u_2 = -7 \int \sin x \, dx = 7 \cos x$$

$$\therefore y_p = u_1 y_1 + u_2 y_2 = (-7x \cos x + 7 \sin x) \cdot x^2 + 7 \cos x \cdot x^3 = 7x^2 \sin x$$

故可得通解為  $y = y_h + y_p = C_1 x^2 + C_2 x^3 + 7x^2 \sin x$

$$6. xy'' - y' = 2x^2e^x \Rightarrow x^2y'' - xy' = 2x^3e^x$$

令  $y = x^m$  代入 ODE 可得特徵方程式

$$m(m-1) - m = 0 \Rightarrow m^2 - 2m = 0 \Rightarrow m(m-2) = 0 \Rightarrow m = 0, 2$$

$$\therefore y_h = C_1 + C_2x^2$$

$$\text{令 } y_p = u_1y_1 + u_2y_2$$

$$\text{可知 } W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} 1 & x^2 \\ 0 & 2x \end{vmatrix} = 2x$$

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{W} = \frac{\begin{vmatrix} 0 & x^2 \\ 2xe^x & 2x \end{vmatrix}}{2x} = -x^2e^x \Rightarrow u_1 = -\int x^2e^x dx = -e^x(2 - 2x + x^2)$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{W} = \frac{\begin{vmatrix} 1 & 0 \\ 0 & 2xe^x \end{vmatrix}}{2x} = e^x \Rightarrow u_2 = \int e^x dx = e^x$$

$$\therefore y_p = u_1y_1 + u_2y_2 = -e^x(2 - 2x + x^2) \cdot 1 + e^x \cdot x^2 = e^x(2x - 2)$$

$$\text{故可得通解為 } y = y_h + y_p = C_1 + C_2x^2 + e^x(2x - 2)$$

$$7. x^2y'' + xy' + y = \sec(\ln x)$$

令  $y = x^m$  代入 ODE 可得特徵方程式

$$m(m-1) + m + 1 = 0 \Rightarrow m^2 + 1 = 0 \Rightarrow m = \pm i$$

$$\therefore y_h = C_1 \cos(\ln x) + C_2 \sin(\ln x)$$

$$\text{令 } y_p = u_1y_1 + u_2y_2$$

$$\text{可知 } W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos(\ln x) & \sin(\ln x) \\ -\frac{1}{x}\sin(\ln x) & \frac{1}{x}\cos(\ln x) \end{vmatrix} = x^{-1}$$

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{W} = \frac{\begin{vmatrix} 0 & \sin(\ln x) \\ x^{-2}\sec(\ln x) & x^{-1}\cos(\ln x) \end{vmatrix}}{x^{-1}} = -\frac{1}{x}\sec(\ln x)\sin(\ln x)$$

$$\begin{aligned} \Rightarrow u_1 &= -\int \frac{1}{x}\sec(\ln x)\sin(\ln x) dx = -\int \frac{\sin(\ln x)}{\cos(\ln x)} d(\ln x) = \int \frac{1}{\cos(\ln x)} d(\cos(\ln x)) \\ &= \ln(\cos(\ln x)) \end{aligned}$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & \frac{f(x)}{x^2} \end{vmatrix}}{W} = \frac{\begin{vmatrix} \cos(\ln x) & 0 \\ -\frac{1}{x} \sin(\ln x) & x^{-2} \sec(\ln x) \end{vmatrix}}{x^{-1}} = x^{-1} \Rightarrow u_2 = \int \frac{1}{x} dx = \ln x$$

$$\therefore y_p = u_1 y_1 + u_2 y_2 = \ln |\cos(\ln x)| \cdot \cos(\ln x) + \ln x \cdot \sin(\ln x)$$

故可得通解為

$$y = y_h + y_p = C_1 \cos(\ln x) + C_2 \sin(\ln x) + \ln |\cos(\ln x)| \cdot \cos(\ln x) + \ln x \cdot \sin(\ln x)$$