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1. 試以待定係數法求解下述微分方程式：

- (1)  $y'' - 5y' + 6y = x^3$
- (2)  $y'' - 2y' = 3x$
- (3)  $y'' - 6y' + 9y = 8e^{3x}$
- (4)  $y'' + 4y = \sin 2x$
- (5)  $y'' + 3y' + 2y = e^{2x} \cos x$
- (6)  $y'' - 2y' + 2y = 2e^x \sin x$
- (7)  $y'' + 2y' + y = xe^{2x}$

2. 試以參數變異法求解下述微分方程式：

- (1)  $y'' - 4y = 8x^2 - 2x$
- (2)  $y'' + 4y = \tan 2x$
- (3)  $y'' - 6y' + 9y = x^{-2}e^{3x}$

**參考解答：**

1. (1)  $y'' - 5y' + 6y = x^3$

令  $y = e^{\lambda x}$  代回 ODE 可得

$$(\lambda^2 - 5\lambda + 6)e^{\lambda x} = 0 \Rightarrow \lambda^2 - 5\lambda + 6 = 0 \Rightarrow (\lambda - 2)(\lambda - 3) = 0 \Rightarrow \lambda = 2, 3$$

$$\therefore y_h = C_1 e^{2x} + C_2 e^{3x}$$

令  $y_p = Ax^3 + Bx^2 + Cx + D$  代回 ODE 可得

$$\begin{aligned} (6Ax + 2B) - 5(3Ax^2 + 2Bx + C) + 6(Ax^3 + Bx^2 + Cx + D) &= x^3 \\ \Rightarrow 6Ax^3 + (6B - 15A)x^2 + (6C - 10B + 6A)x + (6D - 5C + 2B) &= x^3 \end{aligned}$$

$$\text{比較係數可得： } A = \frac{1}{6}, \quad B = \frac{5}{12}, \quad C = \frac{19}{36}, \quad D = \frac{65}{216}$$

$$\therefore y_p = \frac{1}{6}x^3 + \frac{5}{12}x^2 + \frac{19}{36}x + \frac{65}{216}$$

$$y = y_h + y_p = C_1 e^{2x} + C_2 e^{3x} + \frac{1}{6}x^3 + \frac{5}{12}x^2 + \frac{19}{36}x + \frac{65}{216}$$

(2)  $y'' - 2y' = 3x$

令  $y = e^{\lambda x}$  代回 ODE 可得

$$(\lambda^2 - 2\lambda)e^{\lambda x} = 0 \Rightarrow \lambda^2 - 2\lambda = 0 \Rightarrow \lambda(\lambda - 2) = 0 \Rightarrow \lambda = 0, 2$$

$$\therefore y_h = C_1 + C_2 e^{2x}$$

令  $y_p = x(Ax + B)$  代回 ODE 可得

$$2A - 2(2Ax + B) = 3x \Rightarrow -4Ax + (2A - 2B) = 3x$$

$$\text{比較係數可得: } A = -\frac{3}{4}, \quad B = -\frac{3}{4}$$

$$\therefore y_p = -\frac{3}{4}x^2 - \frac{3}{4}x$$

$$y = y_h + y_p = C_1 + C_2 e^{2x} - \frac{3}{4}x^2 - \frac{3}{4}x$$

$$(3) \quad y'' - 6y' + 9y = 8e^{3x}$$

令  $y = e^{\lambda x}$  代回 ODE 可得

$$(\lambda^2 - 6\lambda + 9)e^{\lambda x} = 0 \Rightarrow \lambda^2 - 6\lambda + 9 = 0 \Rightarrow (\lambda - 3)^2 = 0 \Rightarrow \lambda = 3, 3$$

$$\therefore y_h = C_1 e^{3x} + C_2 x e^{3x}$$

$$\text{令 } y_p = Ax^2 e^{3x} \Rightarrow y'_p = A(2x + 3x^2)e^{3x}$$

$$\Rightarrow y''_p = A(2 + 12x + 9x^2)e^{3x} \quad \text{代回 ODE 可得}$$

$$A(2 + 12x + 9x^2)e^{3x} - 6A(2x + 3x^2)e^{3x} + 9Ax^2 e^{3x} = 8e^{3x} \Rightarrow 2A = 8 \Rightarrow A = 4$$

$$\therefore y_p = 4x^2 e^{3x}$$

$$y = y_h + y_p = C_1 e^{3x} + C_2 x e^{3x} + 4x^2 e^{3x}$$

$$(4) \quad y'' + 4y = \sin 2x$$

令  $y = e^{\lambda x}$  代回 ODE 可得

$$(\lambda^2 + 4)e^{\lambda x} = 0 \Rightarrow \lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i$$

$$\therefore y_h = C_1 \cos 2x + C_2 \sin 2x$$

$$\text{令 } y_p = x(A \cos 2x + B \sin 2x) = x \cdot y_h \Rightarrow y'_p = y_h + x \cdot y'_h$$

$$\Rightarrow y''_p = 2y'_h + x \cdot y''_h \quad \text{代回 ODE 可得}$$

$$2y'_h + x \cdot y''_h + 4x \cdot y_h = \sin 2x \Rightarrow 2y'_h = \sin 2x$$

$$\Rightarrow 2(-2A \sin 2x + 2B \cos 2x) = \sin 2x$$

$$\Rightarrow A = -\frac{1}{4}, \quad B = 0$$

$$\therefore y_p = -\frac{1}{4}x \cos 2x$$

$$y = y_h + y_p = C_1 \cos 2x + C_2 \sin 2x - \frac{1}{4} x \cos 2x$$

(5)  $y'' + 3y' + 2y = e^{2x} \cos x$

令  $y = e^{\lambda x}$  代回 ODE 可得

$$(\lambda^2 + 3\lambda + 2)e^{\lambda x} = 0 \Rightarrow \lambda^2 + 3\lambda + 2 = 0 \Rightarrow (\lambda + 1)(\lambda + 2) = 0$$

$$\Rightarrow \lambda = -1, -2$$

$$\therefore y_h = C_1 e^{-x} + C_2 e^{-2x}$$

令  $y_p = e^{2x}(A \cos x + B \sin x)$

$$\Rightarrow y_p' = 2e^{2x}(A \cos x + B \sin x) + e^{2x}(-A \sin x + B \cos x)$$

$$\Rightarrow y_p'' = 3e^{2x}(A \cos x + B \sin x) + 4e^{2x}(-A \sin x + B \cos x) \quad \text{代回 ODE 可得}$$

$$3e^{2x}(A \cos x + B \sin x) + 4e^{2x}(-A \sin x + B \cos x)$$

$$+ 6e^{2x}(A \cos x + B \sin x) + 3e^{2x}(-A \sin x + B \cos x) + 2e^{2x}(A \cos x + B \sin x)$$

$$= e^{2x} \cos x$$

$$\Rightarrow 11e^{2x}(A \cos x + B \sin x) + 7e^{2x}(-A \sin x + B \cos x) = e^{2x} \cos x$$

$$\Rightarrow \begin{cases} 11A + 7B = 1 \\ -7A + 11B = 0 \end{cases}$$

$$\Rightarrow A = \frac{11}{170}, B = \frac{7}{170}$$

$$\therefore y_p = e^{2x} \left( \frac{11}{170} \cos x + \frac{7}{170} \sin x \right)$$

$$y = y_h + y_p = C_1 e^{-x} + C_2 e^{-2x} + e^{2x} \left( \frac{11}{170} \cos x + \frac{7}{170} \sin x \right)$$

(6)  $y'' - 2y' + 2y = 2e^x \sin x$

令  $y = e^{\lambda x}$  代回 ODE 可得

$$(\lambda^2 - 2\lambda + 2)e^{\lambda x} = 0 \Rightarrow \lambda^2 - 2\lambda + 2 = 0 \Rightarrow \lambda = \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 2}}{2} = 1 \pm i$$

$$\therefore y_h = e^x (C_1 \cos x + C_2 \sin x)$$

令  $y_p = x \cdot e^x (A \cos x + B \sin x) = x \cdot y_h$

$$\Rightarrow y_p' = y_h + x \cdot y_h'$$

$$\Rightarrow y_p'' = y_h' + y_h' + x \cdot y_h'' = 2y_h' + x \cdot y_h'' \quad \text{代回 ODE 可得}$$

$$(2y'_h + x \cdot y''_h) - 2(y_h + x \cdot y'_h) + 2(x \cdot y_h) = 2e^x \sin x$$

$$\Rightarrow y'_h - y_h = e^x \sin x$$

$$\Rightarrow e^x(A \cos x + B \sin x) + e^x(-A \sin x + B \cos x) - e^x(A \cos x + B \sin x) = e^x \sin x$$

$$\Rightarrow A = -1, B = 0$$

$$\therefore y_p = -x \cdot e^x \cos x$$

$$y = y_h + y_p = e^x(C_1 \cos x + C_2 \sin x) - x \cdot e^x \cos x$$

$$(7) \quad y'' + 2y' + y = xe^{2x}$$

令  $y = e^{\lambda x}$  代回 ODE 可得

$$(\lambda^2 + 2\lambda + 1)e^{\lambda x} = 0 \Rightarrow \lambda^2 + 2\lambda + 1 = 0 \Rightarrow (\lambda + 1)^2 = 0 \Rightarrow \lambda = -1, -1$$

$$\therefore y_h = e^{-x}(C_1 + C_2 x)$$

$$\text{令 } y_p = (Ax + B)e^{2x}$$

$$\Rightarrow y'_p = (2Ax + 2B + A)e^{2x}$$

$$\Rightarrow y''_p = (4Ax + 4B + 4A)e^{2x} \quad \text{代回 ODE 可得}$$

$$(4Ax + 4B + 4A)e^{2x} + 2(2Ax + 2B + A)e^{2x} + (Ax + B)e^{2x} = xe^{2x}$$

$$\Rightarrow 9Ax + 9B + 6A = x$$

$$\Rightarrow A = \frac{1}{9}, B = -\frac{2}{27}$$

$$\therefore y_p = \left(\frac{1}{9}x - \frac{2}{27}\right)e^{2x}$$

$$y = y_h + y_p = e^{-x}(C_1 + C_2 x) + \left(\frac{1}{9}x - \frac{2}{27}\right)e^{2x}$$

$$2. (1) \quad y'' - 4y = 8x^2 - 2x$$

令  $y = e^{\lambda x}$  代入 ODE 可得特徵方程式

$$\lambda^2 - 4 = 0 \Rightarrow (\lambda + 2)(\lambda - 2) = 0 \Rightarrow \lambda = -2, 2$$

$$\therefore y_h = C_1 e^{-2x} + C_2 e^{2x}$$

$$\text{令 } y_p = u_1 y_1 + u_2 y_2$$

$$\text{可知 } W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} e^{-2x} & e^{2x} \\ -2e^{-2x} & 2e^{2x} \end{vmatrix} = 4$$

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{W} = \frac{\begin{vmatrix} 0 & e^{2x} \\ 8x^2 - 2x & 2e^{2x} \end{vmatrix}}{4} = -(2x^2 - \frac{1}{2}x)e^{2x} \Rightarrow u_1 = -(x^2 - \frac{5}{4}x + \frac{5}{8})e^{2x}$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{W} = \frac{\begin{vmatrix} e^{-2x} & 0 \\ -2e^{-2x} & 8x^2 - 2x \end{vmatrix}}{4} = (2x^2 - \frac{1}{2}x)e^{-2x} \Rightarrow u_2 = -(x^2 + \frac{3}{4}x + \frac{3}{8})e^{-2x}$$

$$\begin{aligned} \therefore y_p &= u_1 y_1 + u_2 y_2 = -(x^2 - \frac{5}{4}x + \frac{5}{8})e^{2x} \cdot e^{-2x} - (x^2 + \frac{3}{4}x + \frac{3}{8})e^{-2x} \cdot e^{2x} \\ &= -2x^2 + \frac{1}{2}x - 1 \end{aligned}$$

故可得通解為  $y = y_h + y_p = C_1 e^{-2x} + C_2 e^{2x} - 2x^2 + \frac{1}{2}x - 1$

(2)  $y'' + 4y = \tan 2x$

令  $y = e^{\lambda x}$  代入 ODE 可得特徵方程式

$$\lambda^2 + 4 = 0 \Rightarrow \lambda^2 = -4 \Rightarrow \lambda = \pm 2i$$

$$\therefore y_h = C_1 \cos 2x + C_2 \sin 2x$$

令  $y_p = u_1 y_1 + u_2 y_2$

可知  $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{W} = \frac{\begin{vmatrix} 0 & \sin 2x \\ \tan 2x & 2\cos 2x \end{vmatrix}}{2} = -\frac{1}{2} \tan 2x \cdot \sin 2x$$

$$\Rightarrow u_1 = \frac{1}{4} (\ln|\cos x - \sin x| - \ln|\cos x + \sin x| + \sin 2x)$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{W} = \frac{\begin{vmatrix} \cos 2x & 0 \\ -2\sin 2x & \tan 2x \end{vmatrix}}{2} = \frac{1}{2} \sin 2x \Rightarrow u_2 = -\frac{1}{4} \cos 2x$$

$$\therefore y_p = u_1 y_1 + u_2 y_2$$

$$= \frac{1}{4} (\ln|\cos x - \sin x| - \ln|\cos x + \sin x| + \sin 2x) \cdot \cos 2x - \frac{1}{4} \cos 2x \cdot \sin 2x$$

$$= \frac{1}{4} (\ln|\cos x - \sin x| - \ln|\cos x + \sin x|) \cdot \cos 2x$$

$$\begin{aligned}
&= -\frac{1}{4} \cos 2x \cdot \ln \left| \frac{\cos x + \sin x}{\cos x - \sin x} \right| \\
&= -\frac{1}{4} \cos 2x \cdot \ln \left| \frac{1 + \tan x}{1 - \tan x} \right| \\
&= -\frac{1}{4} \cos 2x \cdot \ln \left| \tan \left( x + \frac{\pi}{4} \right) \right| \quad \left( \tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} \right)
\end{aligned}$$

故可得通解為  $y = y_h + y_p = C_1 \cos 2x + C_2 \sin 2x - \frac{1}{4} \cos 2x \cdot \ln \left| \tan \left( \frac{\pi}{4} + x \right) \right|$

(3)  $y'' - 6y' + 9y = x^{-2}e^{3x}$

令  $y = e^{\lambda x}$  代入 ODE 可得特徵方程式

$$\lambda^2 - 6\lambda + 9 = 0 \Rightarrow (\lambda - 3)^2 = 0 \Rightarrow \lambda = 3, 3$$

$$\therefore y_h = C_1 e^{3x} + C_2 x e^{3x}$$

令  $y_p = u_1 y_1 + u_2 y_2$

可知  $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & x e^{3x} \\ 3e^{3x} & (3x+1)e^{3x} \end{vmatrix} = e^{6x}$

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{W} = \frac{\begin{vmatrix} 0 & x e^{3x} \\ x^{-2} e^{3x} & (3x+1)e^{3x} \end{vmatrix}}{e^{6x}} = -x^{-1} \Rightarrow u_1 = -\ln|x|$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{W} = \frac{\begin{vmatrix} e^{3x} & 0 \\ 3e^{3x} & x^{-2} e^{3x} \end{vmatrix}}{e^{6x}} = x^{-2} \Rightarrow u_2 = -\frac{1}{x}$$

$$\therefore y_p = u_1 y_1 + u_2 y_2 = -\ln|x| \cdot e^{3x} - \frac{1}{x} \cdot x e^{3x} = -\ln|x| \cdot e^{3x} - e^{3x}$$

故可得通解為  $y = y_h + y_p = e^{3x}(C_1 + C_2 x - \ln|x|)$