

系級：_____ 學號：_____ 姓名：_____

- 給一微分方程 $y''+6y'+9y=0$ ，已知其解為 $y_1 = e^{-3x}$ (重根)，試用已知一解求另一補解找出 $y_2 = ?$
- 給一微分方程 $x^2y''-3xy'+4y=0$ ，已知一補解為 $y_1 = x^2$ ，試求另一補解 y_2
- 試解下述二階以上常係數齊次 ODE
 - $y''-4y'+2y=0$
 - $y''-10y'+25y=0$
 - $y''+2y'+6y=0$
 - $y''-2y'+(\pi^2+1)y=0$; $y(\frac{1}{4})=0$, $y'(\frac{1}{4})=-\pi\sqrt[4]{4e}$
 - $4y''+4y'+1y=0$; $y(0)=-2$, $y(2)=e^{-1}$
 - $y''' + 6y'' + 12y' + 8y = 0$

<參考解答>

- $y''+6y'+9y=0$ 已知其解為 $y_1 = e^{-3x}$

由 Wronskian 可知

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2 = C_1 e^{-\int adx}$$

$$\Rightarrow y_2' - \frac{y_1'}{y_1} y_2 = \frac{C_1}{y_1} e^{-\int adx} = \frac{C_1}{y_1} e^{-\int 6dx} = \frac{C_1}{y_1} e^{-6x}$$

又 $y_1 = e^{-3x}$ 代入後可得

$$\therefore y_2' + 3y_2 = C_1 e^{-3x} \longrightarrow \text{此為一階線性 ODE}$$

可知積分因子為 $\mu = e^{\int p(x)dx} = e^{\int 3dx} = e^{3x}$

$$e^{3x} y_2' + 3e^{3x} y_2 = C_1 \Rightarrow \frac{d}{dx}(e^{3x} y_2) = C_1$$

$$\Rightarrow e^{3x} y_2 = C_1 x + C_2$$

$$\Rightarrow y_2 = C_1 x e^{-3x} + C_2 e^{-3x}$$

$y_1 = e^{-3x}$ 為其一補解，故可知另一補解為 $y_2 = x e^{-3x}$

- $x^2y''-3xy'+4y=0 \Rightarrow y'' - \frac{3}{x}y' + \frac{4}{x^2}y=0$ 已知其解為 $y_1 = x^2$

由 Wronskian 可知

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2 = C_1 e^{-\int adx}$$

$$\Rightarrow y_2' - \frac{y_1'}{y_1} y_2 = \frac{C_1}{y_1} e^{-\int adx} = \frac{C_1}{y_1} e^{\int \frac{3}{x} dx} = \frac{C_1}{y_1} e^{3 \ln x} = \frac{C_1}{y_1} x^3$$

又 $y_1 = x^2$ 代入後可得

$$\therefore y_2' - \frac{2}{x} y_2 = C_1 x \longrightarrow \text{此為一階線性 ODE}$$

$$\text{可知積分因子為 } \mu = e^{\int p(x) dx} = e^{-\int \frac{2}{x} dx} = e^{-2 \ln |x|} = \frac{1}{x^2}$$

$$\frac{1}{x^2} y_2' - \frac{2}{x^3} y_2 = \frac{C_1}{x} \Rightarrow \frac{d}{dx} \left(\frac{1}{x^2} y_2 \right) = \frac{C_1}{x}$$

$$\Rightarrow \frac{1}{x^2} y_2 = C_1 \ln x + C_2$$

$$\Rightarrow y_2 = C_1 x^2 \ln x + C_2 x^2$$

$y_1 = x^2$ 為其一補解，故可知另一補解為 $y_2 = x^2 \ln x$

3. (1) $y'' - 4y' + 2y = 0$

令 $y = e^{\lambda x}$ 代回 ODE 可得

$$(\lambda^2 - 4\lambda + 2)e^{\lambda x} = 0$$

$$\Rightarrow \lambda^2 - 4\lambda + 2 = 0$$

$$\Rightarrow \lambda = \frac{4 \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 2}}{2} = 2 \pm \sqrt{2}$$

$$\therefore y = C_1 e^{(2+\sqrt{2})x} + C_2 e^{(2-\sqrt{2})x}$$

(2) $y'' - 10y' + 25y = 0$

令 $y = e^{\lambda x}$ 代回 ODE 可得

$$(\lambda^2 - 10\lambda + 25)e^{\lambda x} = 0$$

$$\Rightarrow \lambda^2 - 10\lambda + 25 = 0$$

$$\Rightarrow (\lambda - 5)^2 = 0$$

$$\Rightarrow \lambda = 5, 5 \text{ (重根)}$$

$$\therefore y = C_1 e^{5x} + C_2 x e^{5x}$$

(3) $y'' + 2y' + 6y = 0$

令 $y = e^{\lambda x}$ 代回 ODE 可得

$$(\lambda^2 + 2\lambda + 6)e^{\lambda x} = 0$$

$$\Rightarrow \lambda^2 + 2\lambda + 6 = 0$$

$$\Rightarrow \lambda = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 6}}{2} = -1 \pm \sqrt{5} i$$

$$\therefore y = e^{-x} (C_1 \cos \sqrt{5} x + C_2 \sin \sqrt{5} x)$$

$$(4) \quad y'' - 2y' + (\pi^2 + 1)y = 0$$

令 $y = e^{\lambda x}$ 代回 ODE 可得

$$[\lambda^2 - 2\lambda + (\pi^2 + 1)]e^{\lambda x} = 0$$

$$\Rightarrow \lambda^2 - 2\lambda + (\pi^2 + 1) = 0$$

$$\Rightarrow \lambda = \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot (\pi^2 + 1)}}{2} = 1 \pm \pi i$$

$$\therefore y = e^x (C_1 \cos \pi x + C_2 \sin \pi x)$$

$$\text{又 } y\left(\frac{1}{4}\right) = 0 \Rightarrow e^{\frac{1}{4}} (C_1 \cos \frac{\pi}{4} + C_2 \sin \frac{\pi}{4}) = 0 \Rightarrow C_1 + C_2 = 0 \Rightarrow C_1 = -C_2 \quad \dots(1)$$

$$y'\left(\frac{1}{4}\right) = -\pi \sqrt[4]{4e}$$

$$\Rightarrow e^{\frac{1}{4}} (C_1 \cos \frac{\pi}{4} + C_2 \sin \frac{\pi}{4}) + \pi e^{\frac{1}{4}} (-C_1 \sin \frac{\pi}{4} + C_2 \cos \frac{\pi}{4}) = -\pi \sqrt[4]{4e}$$

$$\Rightarrow C_1(1 - \pi) + C_2(1 + \pi) = -2\pi \quad \dots(2)$$

將(1)代入(2)可得 $C_2 = -1$ 代回(1)可得 $C_1 = 1$

$$\therefore y = e^x (\cos \pi x - \sin \pi x)$$

$$(5) \quad 4y'' + 4y' + 1y = 0$$

令 $y = e^{\lambda x}$ 代回 ODE 可得

$$(4\lambda^2 + 4\lambda + 1)e^{\lambda x} = 0$$

$$\Rightarrow 4\lambda^2 + 4\lambda + 1 = 0$$

$$\Rightarrow \lambda = -\frac{1}{2}, -\frac{1}{2} \quad (\text{重根})$$

$$\therefore y = e^{-\frac{x}{2}} (C_1 + C_2 x)$$

$$\text{又 } y(0) = -2 \Rightarrow C_1 = -2$$

$$y(2) = e^{-1} \Rightarrow C_2 = \frac{3}{2}$$

$$\therefore y = e^{-\frac{x}{2}} \left(-2 + \frac{3}{2}x\right)$$

$$(6) \quad y''' + 6y'' + 12y' + 8y = 0$$

令 $y = e^{\lambda x}$ 代回 ODE 可得

$$(\lambda^3 + 6\lambda^2 + 12\lambda + 8)e^{\lambda x} = 0$$

$$\Rightarrow \lambda^3 + 6\lambda^2 + 12\lambda + 8 = 0$$

$$\Rightarrow (\lambda + 2)^3 = 0$$

$$\Rightarrow \lambda = -2, -2, -2 \quad (\text{三重根})$$

$$\therefore y = e^{-2x} (C_1 + C_2 x + C_3 x^2)$$