國立臺灣海洋大學河海工程學系 2020 工程數學(一) 2A 班第四次作業參考解答

系級：________　　學號：________　　姓名：________

1. 試解下述一階高次微分方程式:
   (1) $x^2(y')^2+4xyy'+3y^2=0$
   (2) $xy(y')^2+(x+y)y'+1=0$
   (3) $(y')^4-(x+2y+1)(y')^3+(x+2y+2xy)(y')^2-2xyy'=0$
   (4) $y=xy'-\dfrac{1}{y'}$
2. 試以全微分法求解下述微分方程
   (1) $(2x^2y+3y^3)dx-(x^3+2xy^2)dy=0$
   (2) $y'=\dfrac{y+xy^3(1+\ln x)}{x}$
   (3) $(3xe^y+2y)dx+(x^2e^y+x)dy=0$
   (4) $y'=\dfrac{y+\sqrt{x^2+y^2}}{x}$

3. 試以皮卡德(Picard)法求解:
   $y'=1+y^2$，$y(0)=0$

參考解答:

1.

(1) $x^2(y')^2+4xyy'+3y^2=0$

令 $u=y'$ 代回 ODE 可得 $x^2u^2+4xyu+3y^2=0$

$$\Rightarrow (xu+3y)(xu+y)=0$$

$$\Rightarrow xu+3y=0 \quad \text{or} \quad xu+y=0$$

(a) $xu+3y=0 \quad \Rightarrow y'=-\dfrac{3}{x}y \quad \Rightarrow \dfrac{1}{y}dy=-\dfrac{3}{x}dx$

$$\Rightarrow \int\frac{1}{y}dy=-\int\frac{3}{x}dx$$

$$\Rightarrow \ln|y|=-3\ln|x|+\ln|C|$$

$$\Rightarrow y=\frac{C}{x^3}$$

(b) $xu+y=0 \quad \Rightarrow y'=-\dfrac{1}{x}y \quad \Rightarrow \dfrac{1}{y}dy=-\dfrac{1}{x}dx$

$$\Rightarrow \int\frac{1}{y}dy=-\int\frac{1}{x}dx$$

$$\Rightarrow \ln|y| = -\ln|x| + \ln|C|$$

$$\Rightarrow y = \frac{C}{x}$$

∴ 此 ODE 的解為 $(y - \frac{C}{x^3})(y - \frac{C}{x}) = 0$

(2) $xy(y')^2 + (x+y)y' + 1 = 0$

令 $u = y'$ 代回 ODE 可得 $xyu^2 + (x+y)u + 1 = 0$

$$\Rightarrow (xu+1)(yu+1) = 0$$

$$\Rightarrow xu + 1 = 0 \quad \text{or} \quad yu + 1 = 0$$

(a) $xu + 1 = 0 \quad \Rightarrow y' = -\frac{1}{x} \quad \Rightarrow y = -\ln|x| + C \quad \Rightarrow x = Ce^{-y}$

(b) $yu + 1 = 0 \quad \Rightarrow yy' = -1 \quad \Rightarrow \int y dy = -\int dx$

$$\Rightarrow \frac{1}{2} y^2 = -x + C$$

∴ 此 ODE 的解為 $(x - Ce^{-y})(x + \frac{1}{2} y^2 - C) = 0$

(3) $(y')^4 - (x+2y+1)(y')^3 + (x+2y+2xy)(y')^2 - 2xyy' = 0$

令 $u = y'$ 代回 ODE 可得 $u^4 - (x+2y+1)u^3 + (x+2y+2xy)u^2 - 2xyu = 0$

$$\Rightarrow u(u-x)(u-2y)(u-1) = 0$$

$$\Rightarrow u = 0,\ u - x = 0,\ u - 2y = 0 \ \text{or} \ u - 1 = 0$$

(a) $u = 0 \quad \Rightarrow y' = 0 \quad \Rightarrow y = C$

(b) $u - x = 0 \quad \Rightarrow y' = x \quad \Rightarrow y = \frac{1}{2} x^2 + C$

(c) $u - 2y = 0 \quad \Rightarrow y' = 2y \quad \Rightarrow \ln|y| = 2x + C \quad \Rightarrow y = Ce^{2x}$

(b) $u - 1 = 0 \quad \Rightarrow y' = 1 \quad \Rightarrow y = x + C$

∴ 此 ODE 的解為 $(y - C)(y - \frac{1}{2}x^2 - C)(y - Ce^{2x})(y - x - C) = 0$

(4) $y = xy' - \frac{1}{y'}$

令 $u = y'$ 代回 ODE 可得 $y = xu - \frac{1}{u}$

同時將兩邊對 x 微分: $y' = u + xu' + \frac{u'}{u^2} \quad \Rightarrow u = u + xu' + \frac{u'}{u^2}$

$$\Rightarrow u'(x+\frac{1}{u^2})=0$$

$$\Rightarrow u'=0 \quad 或 \quad u^2=-\frac{1}{x}$$

當 $u'=0 \quad \Rightarrow u=c \quad \Rightarrow y=cx-\frac{1}{c}$ （通解）

當 $u^2=-\frac{1}{x} \quad \Rightarrow uy=xu^2-1 \quad \Rightarrow uy=-2 \quad \Rightarrow u^2y^2=4 \quad \Rightarrow y^2=-4x$ （奇解）

2.

(1) $(2x^2y+3y^3)dx-(x^3+2xy^2)dy=0$

$$\Rightarrow (2x^2ydx-x^3dy)+(3y^3dx-2xy^2dy)=0$$

$$\Rightarrow x^2(2ydx-xdy)+y^2(3ydx-2xdy)=0$$

$$\Rightarrow x^2\frac{d(x^2y^{-1})}{xy^{-2}}+y^2\frac{d(x^3y^{-2})}{x^2y^{-3}}=0$$

$$\Rightarrow xy^2d(x^2y^{-1})+x^{-2}y^5d(x^3y^{-2})=0$$

$$\Rightarrow x^{-6}y^3d(x^2y^{-1})+x^{-9}y^6d(x^3y^{-2})=0 \qquad (同乘\ x^{-7}y)$$

$$\Rightarrow -\frac{1}{2}(x^2y^{-1})^{-2}-\frac{1}{2}(x^3y^{-2})^{-2}=C$$

(2) $y'=\frac{y+xy^3(1+\ln x)}{x} \qquad \Rightarrow xdy-ydx=xy^3(1+\ln x)dx$

$$\Rightarrow \frac{x}{y}\frac{xdy-ydx}{y^2}=x^2(1+\ln x)dx$$

$$\Rightarrow -\frac{x}{y}d(\frac{x}{y})=x^2(1+\ln x)dx$$

$$\Rightarrow -\int\frac{x}{y}d(\frac{x}{y})=\int x^2(1+\ln x)dx$$

$$\Rightarrow -\frac{1}{2}(\frac{x}{y})^2=\frac{1}{3}x^3+\frac{1}{3}x^2\ln x-\frac{1}{9}x^3+C$$

$$\Rightarrow -\frac{1}{2}(\frac{x}{y})^2=\frac{2}{3}x^3+\frac{1}{3}x^2\ln x+C$$

(3) $(3xe^y+2y)dx+(x^2e^y+x)dy=0$

$$\Rightarrow (3xe^ydx+x^2e^ydy)+(2ydx+xdy)=0$$

$$\Rightarrow \frac{d(x^3e^y)}{x}+\frac{d(x^2y)}{x}=0$$

$$\Rightarrow d(x^3e^y)+d(x^2y)=0$$

$$\Rightarrow x^3e^y+x^2y=C$$

(4) $y' = \dfrac{y+\sqrt{x^2+y^2}}{x} \quad \Rightarrow xdy - ydx = \sqrt{x^2+y^2}\ dx$

$$\Rightarrow \frac{xdy - ydx}{x^2} = \frac{1}{x}\sqrt{1+(\frac{y}{x})^2}\ dx$$

$$\Rightarrow \frac{1}{\sqrt{1+(\frac{y}{x})^2}} d(\frac{y}{x}) = \frac{1}{x}\, dx$$

$$\Rightarrow \int \frac{1}{\sqrt{1+(\frac{y}{x})^2}} d(\frac{y}{x}) = \int \frac{1}{x}\, dx$$

令 $\dfrac{y}{x} = \tan\theta \quad \Rightarrow d(\dfrac{y}{x}) = \sec^2\theta\, d\theta$ 代入上式

可得 $\displaystyle\int \sec\theta\, d\theta = \int \frac{1}{x}\, dx \quad \Rightarrow \int \sec\theta \cdot \frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}\, d\theta = \int \frac{1}{x}\, dx$

令 $u = \sec\theta + \tan\theta \quad \Rightarrow du = (\sec\theta\cdot\tan\theta + \sec^2\theta)\, d\theta = \sec\theta(\tan\theta + \sec\theta)\, d\theta$

$\therefore \displaystyle\int \sec\theta \cdot \frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}\, d\theta = \int \frac{1}{u}\, du = \int \frac{1}{x}\, dx$

$$\Rightarrow \ln|u| = \ln|x| + C$$

$$\Rightarrow \ln|\sec\theta + \tan\theta| = \ln|x| + C$$

$$\Rightarrow \ln\left|\sqrt{1+\tan^2\theta} + \tan\theta\right| = \ln|x| + C$$

$$\Rightarrow \ln\left|\sqrt{1+(\frac{y}{x})^2} + \frac{y}{x}\right| = \ln|x| + C$$

3. $y' = 1 + y^2$, $y(0) = 0$

$$y = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt$$

$$y_1 = y_0 + \int_0^x (1+y_0^2)\, dt = 0 + \int_0^x dx = x$$

$$y_2 = y_0 + \int_0^x (1+y_1^2)\, dt = 0 + \int_0^x (1+x^2)\, dx = x + \frac{1}{3}x^3$$

$$y_3 = y_0 + \int_0^x (1+y_2^2)\, dt = 0 + \int_0^x (1 + x^2 + \frac{2}{3}x^4 + \frac{1}{9}x^6)\, dx$$

$$= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{1}{63}x^7$$

$$y_4 = y_0 + \int_0^x (1+y_3^2)\,dt = 0 + \int_0^x [1+(x+\frac{1}{3}x^3+\frac{2}{15}x^5+\frac{1}{63}x^7)^2]\,dx$$

$$= \int_0^x (1+x^2+\frac{2}{3}x^4+\frac{17}{45}x^6+\frac{38}{315}x^8+\frac{131}{4725}x^{10}+\cdots)\,dx$$

$$= x+\frac{1}{3}x^3+\frac{2}{15}x^5+\frac{17}{315}x^7+\frac{38}{2835}x^9+\frac{131}{51975}x^{11}+\cdots$$

$$y_5 = y_0 + \int_0^x (1+y_4^2)\,dt = 0 + \int_0^x [1+(x+\frac{1}{3}x^3+\frac{2}{15}x^5+\frac{17}{315}x^7+\frac{38}{2835}x^9+\cdots)^2]\,dx$$

$$= \int_0^x (1+x^2+\frac{2}{3}x^4+\frac{17}{45}x^6+\frac{62}{315}x^8+\frac{1142}{14175}x^{10}+\cdots)\,dx$$

$$= x+\frac{1}{3}x^3+\frac{2}{15}x^5+\frac{17}{315}x^7+\frac{62}{2835}x^9+\frac{1142}{155925}x^{11}+\cdots$$

$$\vdots$$

$$y_n = x+\frac{1}{3}x^3+\frac{2}{15}x^5+\frac{17}{315}x^7+\frac{62}{2835}x^9+\cdots$$

解析解:

$$y' = 1+y^2 \quad \Rightarrow \frac{1}{1+y^2}dy = dx \quad \Rightarrow \int \frac{1}{1+y^2}dy = \int dx$$

$$\Rightarrow \tan^{-1} y = x + C \quad \Rightarrow y = \tan(x+C)$$

又 $y(0)=0 \quad \Rightarrow C=0$

$\therefore \quad y = \tan x$

由級數展開可知 $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 - \cdots$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \cdots$$

$$y = \tan x = \frac{\sin x}{\cos x} = x+\frac{1}{3}x^3+\frac{2}{15}x^5+\frac{17}{315}x^7+\frac{62}{2835}x^9+\mathrm{O}(x^{11})$$