

系級：_____ 學號：_____ 姓名：_____

試解下述一階線性微分方程式：

1. (1) $xy' + (1+x)y = e^x$
- (2) $(x-2)y' = y + 2(x-2)^3$
- (3) $y' - 2y \cot(2x) = 1 - 2x \cot(2x) - 2 \csc(2x)$
- (4) $y' = 2\frac{y}{x} + x^2 e^x$
- (5) $y' + y = \sin x$

2. (1) $y' - y = xy^5$
- (2) $y' + \frac{3}{x}y = -2xy^{\frac{5}{2}}$

3. (1) $y' - y + e^{-x}y^2 - e^x = 0$
- (2) $y' = xy^2 + (-8x^2 + \frac{1}{x})y + 16x^3, y(2) = 6$

參考解答：

1.

$$(1) \quad xy' + (1+x)y = e^x \quad \Rightarrow \quad y' + \frac{1+x}{x}y = \frac{1}{x}e^x \quad \longrightarrow \quad \text{此為一階線性 ODE}$$

$$\text{積分因子為 } \mu = e^{\int p(x)dx} = e^{\int \frac{1+x}{x}dx} = e^{x+\ln x} = xe^x$$

$$\text{同乘積分因子後可得 } xe^x y' + xe^x \frac{1+x}{x}y = \frac{1}{x}e^x \cdot xe^x$$

$$\Rightarrow \frac{d}{dx}(xe^x y) = e^{2x}$$

$$\Rightarrow xe^x y = \int \frac{1}{x}e^{2x} dx = \frac{1}{2}e^{2x} + C$$

$$\Rightarrow y = \frac{e^x}{2x} + C \frac{e^{-x}}{x}$$

$$(2) \quad (x-2)y' = y + 2(x-2)^3 \quad \Rightarrow \quad y' - \frac{1}{x-2}y = 2(x-2)^2 \quad \longrightarrow \quad \text{此為一階線性 ODE}$$

$$\text{積分因子為 } \mu = e^{\int p(x)dx} = e^{-\int \frac{1}{x-2} dx} = e^{-\ln|x-2|} = \frac{1}{x-2}$$

$$\text{同乘積分因子後可得 } \frac{1}{x-2} y' - \frac{1}{(x-2)^2} y = 2(x-2)$$

$$\Rightarrow \frac{d}{dx} \left(\frac{1}{x-2} y \right) = 2(x-2)$$

$$\Rightarrow \frac{1}{x-2} y = 2 \int (x-2) dx = (x-2)^2 + C$$

$$\Rightarrow y = (x-2)^3 + C(x-2)$$

(3) $y' - 2y \cot(2x) = 1 - 2x \cot(2x) - 2 \csc(2x)$ \longrightarrow 此為一階線性 ODE

$$\text{積分因子為 } \mu = e^{\int p(x)dx} = e^{-2 \int \cot(2x) dx} = e^{-\ln|\sin(2x)|} = \frac{1}{\sin(2x)}$$

同乘積分因子後可得

$$\frac{1}{\sin(2x)} y' - 2 \frac{\cos(2x)}{\sin^2(2x)} y = \frac{1}{\sin(2x)} - 2x \frac{\cos(2x)}{\sin^2(2x)} - 2 \frac{1}{\sin^2(2x)}$$

$$\Rightarrow \frac{d}{dx} \left[\frac{1}{\sin(2x)} y \right] = \frac{1}{\sin(2x)} - 2x \frac{\cos(2x)}{\sin^2(2x)} - 2 \frac{1}{\sin^2(2x)}$$

$$\Rightarrow \frac{1}{\sin(2x)} y = \int \left[\frac{1}{\sin(2x)} - 2x \frac{\cos(2x)}{\sin^2(2x)} - 2 \frac{1}{\sin^2(2x)} \right] dx$$

$$\Rightarrow \frac{1}{\sin(2x)} y = \frac{1}{2} (\ln|\sin x| - \ln|\cos x|) + \frac{x}{\sin 2x} - \frac{1}{2} (\ln|\sin x| - \ln|\cos x|) + \cot(2x) + C$$

$$\Rightarrow y = x + \cos(2x) + C \sin(2x)$$

Note:

$$\int \frac{1}{\sin(2x)} dx = \frac{1}{2} \int \left(\frac{\cos x}{\sin x} + \frac{\sin x}{\cos x} \right) dx = \frac{1}{2} (\ln|\sin x| - \ln|\cos x|)$$

$$\int x \frac{\cos(2x)}{\sin^2(2x)} dx = -\frac{1}{2} \frac{x}{\sin 2x} + \frac{1}{4} (\ln|\sin x| - \ln|\cos x|)$$

$$\int \frac{1}{\sin^2(2x)} dx = \int \frac{\sin^2(2x) + \cos^2(2x)}{\sin^2(2x)} dx = -\frac{1}{2} \frac{\cos(2x)}{\sin(2x)} = -\frac{1}{2} \cot(2x)$$

(4) $y' = 2\frac{y}{x} + x^2 e^x \Rightarrow y' - \frac{2}{x} y = x^2 e^x \longrightarrow$ 此為一階線性 ODE

$$\text{積分因子為 } \mu = e^{\int p(x)dx} = e^{-\int \frac{2}{x} dx} = e^{-2\ln|x|} = \frac{1}{x^2}$$

$$\text{同乘積分因子後可得 } \frac{1}{x^2} y' - \frac{2}{x^3} y = e^x$$

$$\Rightarrow \frac{d}{dx} \left(\frac{1}{x^2} y \right) = e^x$$

$$\Rightarrow \frac{1}{x^2} y = \int e^x dx = e^x + C$$

$$\Rightarrow y = x^2 e^x + Cx^2$$

(5) $y' + y = \sin x$ \longrightarrow 此為一階線性 ODE

$$\text{積分因子為 } \mu = e^{\int p(x)dx} = e^{\int 1dx} = e^x$$

$$\text{同乘積分因子後可得 } e^x y' + e^x y = e^x \sin x$$

$$\Rightarrow \frac{d}{dx}(e^x y) = e^x \sin x$$

$$\Rightarrow e^x y = \int e^x \sin x dx = \frac{1}{2} e^x (\sin x - \cos x) + C$$

$$\Rightarrow y = \frac{1}{2} (\sin x - \cos x) + C e^{-x}$$

2.

(1) $y' - y = xy^5$ \longrightarrow 此為 Bernoulli ODE

$$\Rightarrow y^{-5} y' - y^{-4} = x$$

$$\text{令 } u = y^{-4} \Rightarrow u' = -4y^{-5} y' \text{ 代回 ODE 可得}$$

$$-\frac{1}{4} u' - u = x \Rightarrow u' + 4u = -4x \longrightarrow \text{此為一階線性 ODE}$$

$$\text{積分因子為 } \mu = e^{\int p(x)dx} = e^{\int 4dx} = e^{4x}$$

$$\text{同乘積分因子後可得 } e^{4x} u' + 4e^{4x} u = -4xe^{4x}$$

$$\Rightarrow \frac{d}{dx}(e^{4x} u) = -4xe^{4x}$$

$$\Rightarrow e^{4x} u = -4 \int xe^{4x} dx = e^{4x} \left(\frac{1}{4} - x \right) + C$$

$$\Rightarrow u = \frac{1}{4} - x + C e^{-4x}$$

$$\Rightarrow \frac{1}{y^4} = \frac{1}{4} - x + C e^{-4x}$$

$$\Rightarrow e^{4x} \left(y^{-4} + x - \frac{1}{4} \right) = C$$

(2) $y' + \frac{3}{x} y = -2xy^{\frac{5}{2}}$ \longrightarrow 此為 Bernoulli ODE

$$\Rightarrow y^{-\frac{5}{2}} y' + \frac{3}{x} y^{-\frac{3}{2}} = -2x$$

令 $u = y^{\frac{3}{2}} \Rightarrow u' = -\frac{3}{2}y^{-\frac{5}{2}}y'$ 代回 ODE 可得

$$-\frac{2}{3}u' + \frac{3}{x}u = -2x \Rightarrow u' - \frac{9}{2x}u = 3x \longrightarrow \text{此為一階線性 ODE}$$

$$\text{積分因子為 } \mu = e^{\int p(x)dx} = e^{-\frac{9}{2}\int \frac{1}{x}dx} = e^{-\frac{9}{2}\ln|x|} = x^{-\frac{9}{2}}$$

$$\text{同乘積分因子後可得 } x^{\frac{9}{2}}u' - \frac{9}{2}x^{\frac{11}{2}}u = 3x^{\frac{7}{2}}$$

$$\Rightarrow \frac{d}{dx}(x^{\frac{9}{2}}u) = 3x^{\frac{7}{2}}$$

$$\Rightarrow x^{\frac{9}{2}}u = 3\int x^{\frac{7}{2}}dx = -\frac{6}{5}x^{\frac{5}{2}} + C$$

$$\Rightarrow u = -\frac{6}{5}x^2 + Cx^{\frac{9}{2}}$$

$$\Rightarrow y^{\frac{3}{2}} = -\frac{6}{5}x^2 + Cx^{\frac{9}{2}}$$

3.

(1) $y' - y + e^{-x}y^2 - e^x = 0 \longrightarrow$ 此為 Riccati ODE

由觀察得一解為 $S = e^x$

$$\text{令 } y = S + \frac{1}{V} = e^x + \frac{1}{V} \Rightarrow y' = e^x - \frac{V'}{V^2} \text{ 代回 ODE 可得}$$

$$(e^x - \frac{V'}{V^2}) - (e^x + \frac{1}{V}) + e^{-x}(e^x + \frac{1}{V})^2 - e^x = 0$$

$$\Rightarrow e^x - \frac{V'}{V^2} - e^x - \frac{1}{V} + e^x + \frac{2}{V} + \frac{1}{V^2}e^{-x} - e^x = 0$$

$$\Rightarrow V' - V = e^{-x} \longrightarrow \text{此為一階線性 ODE}$$

$$\text{積分因子為 } \mu = e^{\int p(x)dx} = e^{-\int 1dx} = e^{-x}$$

$$\text{同乘積分因子後可得 } e^{-x}V' - e^{-x}V = e^{-2x}$$

$$\Rightarrow \frac{d}{dx}(e^{-x}V) = e^{-2x}$$

$$\Rightarrow e^{-x}V = \int e^{-2x}dx = -\frac{1}{2}e^{-2x} + C$$

$$\Rightarrow V = -\frac{1}{2}e^{-x} + Ce^x$$

$$\therefore y = S + V = e^x + \frac{1}{-\frac{1}{2}e^{-x} + Ce^x}$$

(2) $y' = xy^2 + (-8x^2 + \frac{1}{x})y + 16x^3 \longrightarrow$ 此為 Riccati ODE

由觀察得一解為 $S = 4x$

令 $y = S + \frac{1}{V} = 4x + \frac{1}{V} \Rightarrow y' = 4 - \frac{V'}{V^2}$ 代回 ODE 可得

$$4 - \frac{V'}{V^2} = x(4x + \frac{1}{V})^2 + (-8x^2 + \frac{1}{x})(4x + \frac{1}{V}) + 16x^3$$

$$\Rightarrow 4 - \frac{V'}{V^2} = 16x^3 + \frac{8}{V}x^2 + \frac{1}{V^2}x - 32x^3 + 4 - \frac{8x^2}{V} + \frac{1}{Vx} + 16x^3$$

$\Rightarrow V' + \frac{1}{x}V = -x \longrightarrow$ 此為一階線性 ODE

積分因子為 $\mu = e^{\int p(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = x$

同乘積分因子後可得 $xV' + V = -x^2$

$$\Rightarrow \frac{d}{dx}(xV) = -x^2$$

$$\Rightarrow xV = -\int x^2 dx = -\frac{1}{3}x^3 + C$$

$$\Rightarrow V = -\frac{1}{3}x^2 + C\frac{1}{x}$$

$$\therefore y = S + V = 4x + \frac{1}{-\frac{1}{3}x^2 + C\frac{1}{x}}$$

$$\text{又 } y(2) = 6 \Rightarrow C = \frac{5}{3}$$

$$\therefore y = 4x + \frac{1}{\frac{5}{3x} - \frac{1}{3}x^2}$$