國立臺灣海洋大學河海工程學系 2020 工程數學(一) 2A 班第二次作業參考解答

系級：________　　學號：________　　姓名：________

試以正合法求下述微分方程式:

1. (1) $(2x^3+3y)dx+(3x+y-1)dy=0$

   (2) $x^3-y\sin x+(\cos x+2y)y'=0$

   (3) $(y^2e^{xy^2}+4x^3)dx+(2xye^{xy^2}-3y^2)dy=0$

   (4) $\dfrac{dy}{dx}=\dfrac{-\cos(xy)+xy\sin(xy)}{-x^2\sin(xy)+2y}$

   (5) $\cos(\pi x)\cos(2\pi y)dx=2\sin(\pi x)\sin(2\pi y)dy,\ y(\frac{3}{2})=\frac{1}{2}$

2. (1) $xdy-ydx-(1-x^2)dx=0$

   (2) $\sin ydx+\cos ydy=0$

   (3) $2dx-e^{y-x}dy=0$

   (4) $(2xy^2+y)dx+(x+2x^2y-x^4y^3)dy=0$

   (5) $(7x^5y^5+2y\sin x+xy\cos x)dx+(6x^6y^4+2x\sin x)dy=0$

參考解答:

1.

(1) $(2x^3+3y)dx+(3x+y-1)dy=0$

令 $M=2x^3+3y \quad \Rightarrow \dfrac{\partial M}{\partial y}=3$

$N=3x+y-1 \quad \Rightarrow \dfrac{\partial N}{\partial x}=3$

由判斷式 $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$ 可知此為正合微分方程

$\therefore\ M=\dfrac{\partial \phi}{\partial x}=2x^3+3y \quad \Rightarrow \phi=\dfrac{1}{2}x^4+3xy+f(y)$

$N=\dfrac{\partial \phi}{\partial y}=3x+y-1 \quad \Rightarrow \phi=3xy+\dfrac{1}{2}y^2-y+g(x)$

由上兩式可知 $\phi(x,y)=\dfrac{1}{2}x^4+3xy+\dfrac{1}{2}y^2-y=C$

(2) $x^3 - y\sin x + (\cos x + 2y)y' = 0 \quad \Rightarrow (x^3 - y\sin x)dx + (\cos x + 2y)dy = 0$

令 $M = x^3 - y\sin x \quad \Rightarrow \dfrac{\partial M}{\partial y} = -\sin x$

$N = \cos x + 2y \quad \Rightarrow \dfrac{\partial N}{\partial x} = -\sin x$

由判斷式 $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ 可知此為正合微分方程

$\therefore\ M = \dfrac{\partial \phi}{\partial x} = x^3 - y\sin x \quad \Rightarrow \phi = \dfrac{1}{4}x^4 + y\cos x + f(y)$

$N = \dfrac{\partial \phi}{\partial y} = \cos x + 2y \quad \Rightarrow \phi = y\cos x + y^2 + g(x)$

由上兩式可知 $\phi(x,y) = \dfrac{1}{4}x^4 + y\cos x + y^2 = C$

(3) $(y^2 e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)dy = 0$

令 $M = y^2 e^{xy^2} + 4x^3 \quad \Rightarrow \dfrac{\partial M}{\partial y} = 2ye^{xy^2} + y^2 e^{xy^2} \cdot 2xy$

$N = 2xye^{xy^2} - 3y^2 \quad \Rightarrow \dfrac{\partial N}{\partial x} = 2ye^{xy^2} + 2xye^{xy^2} \cdot y^2$

由判斷式 $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ 可知此為正合微分方程

$\therefore\ M = \dfrac{\partial \phi}{\partial x} = y^2 e^{xy^2} + 4x^3 \quad \Rightarrow \phi = e^{xy^2} + x^4 + f(y)$

$N = \dfrac{\partial \phi}{\partial y} = 2xye^{xy^2} - 3y^2 \quad \Rightarrow \phi = e^{xy^2} - y^3 + g(x)$

由上兩式可知 $\phi(x,y) = e^{xy^2} + x^4 - y^3 = C$

(4) $\dfrac{dy}{dx} = \dfrac{-\cos(xy) + xy\sin(xy)}{-x^2\sin(xy) + 2y} \Rightarrow [\cos(xy) - xy\sin(xy)]dx + [-x^2\sin(xy) + 2y]dy = 0$

令 $M = \cos(xy) - xy\sin(xy) \quad \Rightarrow \dfrac{\partial M}{\partial y} = -x\sin(xy) - x\sin(xy) - x^2 y\cos(xy)$

$N = -x^2\sin(xy) + 2y \quad \Rightarrow \dfrac{\partial N}{\partial x} = -2x\sin(xy) - x^2 y\cos(xy)$

由判斷式 $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ 可知此為正合微分方程

$\therefore\ M = \dfrac{\partial \phi}{\partial x} = \cos(xy) - xy\sin(xy)$

$$\Rightarrow \phi = \frac{1}{y}\sin(xy) - [-\frac{x}{y}\cos(xy) + \frac{1}{y^2}\sin(xy)]\cdot y + f(y)$$

$$\Rightarrow \phi = x\cos(xy) + f(y)$$

$$N = \frac{\partial \phi}{\partial y} = -x^2\sin(xy) + 2y \qquad \Rightarrow \phi = x\cos(xy) + y^2 + g(x)$$

由上兩式可知 $\phi(x,y) = x\cos(xy) + y^2 = C$

(5) $\cos(\pi x)\cos(2\pi y)dx = 2\sin(\pi x)\sin(2\pi y)dy$

$\Rightarrow \cos(\pi x)\cos(2\pi y)dx - 2\sin(\pi x)\sin(2\pi y)dy = 0$

令 $M = \cos(\pi x)\cos(2\pi y) \qquad \Rightarrow \frac{\partial M}{\partial y} = -2\pi\cos(\pi x)\sin(2\pi y)$

$N = -2\sin(\pi x)\sin(2\pi y) \qquad \Rightarrow \frac{\partial N}{\partial x} = -2\pi\cos(\pi x)\sin(2\pi y)$

由判斷式 $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ 可知此為正合微分方程

$\therefore\ M = \frac{\partial \phi}{\partial x} = \cos(\pi x)\cos(2\pi y) \quad \Rightarrow \phi = \frac{1}{\pi}\sin(\pi x)\cos(2\pi y) + f(y)$

$N = \frac{\partial \phi}{\partial y} = -2\sin(\pi x)\sin(2\pi y) \qquad \Rightarrow \phi = \frac{1}{\pi}\sin(\pi x)\cos(2\pi y) + g(x)$

由上兩式可知 $\phi(x,y) = \frac{1}{\pi}\sin(\pi x)\cos(2\pi y) = C$

又 $y(\frac{3}{2}) = \frac{1}{2} \qquad \Rightarrow C = \frac{1}{\pi}$

$\therefore$ 此 ODE 解為 $\sin(\pi x)\cos(2\pi y) = 1$

2.

(1) $xdy - ydx - (1-x^2)dx = 0 \Rightarrow (1+y-x^2)dx - xdy = 0$

令 $M = 1 + y - x^2 \qquad \Rightarrow \frac{\partial M}{\partial y} = 1$

$N = -x \qquad \Rightarrow \frac{\partial N}{\partial x} = -1$

由判斷式 $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ 可知此非正合微分方程

由 $\frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) = -\frac{2}{x}$ 可知 $\mu = \mu(x)$

故可得 $\int \frac{1}{\mu}d\mu = \int \frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})dx = -\int \frac{2}{x}dx \qquad \Rightarrow \ln|\mu| = -2\ln|x|$

$$\Rightarrow \mu = \frac{1}{x^2}$$

同乘積分因子後可得 $\frac{1+y-x^2}{x^2}dx-\frac{1}{x}dy=0$ 此為正合微分方程

$\therefore\ \overline{M}=\frac{\partial\phi}{\partial x}=\frac{1+y-x^2}{x^2}\quad\Rightarrow\phi=-\frac{1}{x}-\frac{y}{x}-x+f(y)$

$\overline{N}=\frac{\partial\phi}{\partial y}=-\frac{1}{x}\quad\Rightarrow\phi=-\frac{y}{x}+g(x)$

由上兩式可知 $\phi(x,y)=\frac{1}{x}+\frac{y}{x}+x=C$

(2) $\sin y dx+\cos y dy=0$

令 $M=\sin y\quad\Rightarrow\frac{\partial M}{\partial y}=\cos y$

$N=\cos y\quad\Rightarrow\frac{\partial N}{\partial x}=0$

由判斷式 $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$ 可知此非正合微分方程

由 $\frac{1}{N}(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x})=1$ 可知 $\mu=\mu(x)$

故可得 $\int\frac{1}{\mu}d\mu=\int\frac{1}{N}(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x})dx=\int dx\quad\Rightarrow\ln|\mu|=x$

$\Rightarrow\mu=e^x$

同乘積分因子後可得 $e^x\sin y dx+e^x\cos y dy=0$ 此為正合微分方程

$\therefore\ \overline{M}=\frac{\partial\phi}{\partial x}=e^x\sin y\quad\Rightarrow\phi=e^x\sin y+f(y)$

$\overline{N}=\frac{\partial\phi}{\partial y}=e^x\cos y\quad\Rightarrow\phi=e^x\sin y+g(x)$

由上兩式可知 $\phi(x,y)=e^x\sin y=C$

(3) $2dx-e^{y-x}dy=0$

令 $M=2\quad\Rightarrow\frac{\partial M}{\partial y}=0$

$N=-e^{y-x}\quad\Rightarrow\frac{\partial N}{\partial x}=e^{y-x}$

由判斷式 $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$ 可知此非正合微分方程

由 $\frac{1}{N}(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x})=1$ 可知 $\mu=\mu(x)$

故可得 $\int\frac{1}{\mu}d\mu=\int\frac{1}{N}(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x})dx=\int dx\quad\Rightarrow\ln|\mu|=x$

$\Rightarrow\mu=e^x$

同乘積分因子後可得 $2e^{x}dx-e^{y}dy=0$ 此為正合微分方程

$$\therefore\ \overline{M}=\frac{\partial\phi}{\partial x}=2e^{x}\quad\Rightarrow\phi=2e^{x}+f(y)$$

$$\overline{N}=\frac{\partial\phi}{\partial y}=-e^{y}\quad\Rightarrow\phi=-e^{y}+g(x)$$

由上兩式可知 $\phi(x,y)=2e^{x}-e^{y}=C$

(4) $(2xy^{2}+y)dx+(x+2x^{2}y-x^{4}y^{3})dy=0$

令 $M=2xy^{2}+y\quad\Rightarrow\dfrac{\partial M}{\partial y}=2xy+1$

$$N=x+2x^{2}y-x^{4}y^{3}\quad\Rightarrow\frac{\partial N}{\partial x}=1+4xy-4x^{3}y^{3}$$

由判斷式 $\dfrac{\partial M}{\partial y}\neq\dfrac{\partial N}{\partial x}$ 可知此非正合微分方程

由 $\dfrac{1}{N}(\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x})$ 與 $\dfrac{1}{M}(\dfrac{\partial N}{\partial x}-\dfrac{\partial M}{\partial y})$ 可知 μ 非單純為 x 函數或 y 函數

$\therefore$ 令 $\mu=x^{m}y^{n}$

同乘積分因子後可得

$$(2x^{m+1}y^{n+2}+x^{m}y^{n+1})dx+(x^{m+1}y^{n}+2x^{m+2}y^{n+1}-x^{m+4}y^{n+3})dy=0$$

此為正合微分方程

$$\therefore\ \overline{M}=2x^{m+1}y^{n+2}+x^{m}y^{n+1}\quad\Rightarrow\frac{\partial\overline{M}}{\partial y}=2(n+2)x^{m+1}y^{n+1}+(n+1)x^{m}y^{n}$$

$$\overline{N}=x^{m+1}y^{n}+2x^{m+2}y^{n+1}-x^{m+4}y^{n+3}$$

$$\Rightarrow\frac{\partial\overline{N}}{\partial x}=(m+1)x^{m}y^{n}+2(m+2)x^{m+1}y^{n+1}-(m+4)x^{m+3}y^{n+3}$$

又 $\dfrac{\partial\overline{M}}{\partial y}=\dfrac{\partial\overline{N}}{\partial x}\quad\Rightarrow m=-4,\ n=-4$

$$\therefore\ \overline{M}=\frac{\partial\phi}{\partial x}=2x^{-3}y^{-2}+x^{-4}y^{-3}\quad\Rightarrow\phi=-x^{-2}y^{-2}-\frac{1}{3}x^{-3}y^{-3}+f(y)$$

$$\overline{N}=\frac{\partial\phi}{\partial y}=x^{-3}y^{-4}+2x^{-2}y^{-3}-y^{-1}\quad\Rightarrow\phi=-\frac{1}{3}x^{-3}y^{-3}-x^{-2}y^{-2}-\ln|y|+g(x)$$

由上兩式可知 $\phi(x,y)=\dfrac{1}{3}x^{-3}y^{-3}+x^{-2}y^{-2}+\ln|y|=C$

(5) $(7x^{5}y^{5}+2y\sin x+xy\cos x)dx+(6x^{6}y^{4}+2x\sin x)dy=0$

令 $M=7x^{5}y^{5}+2y\sin x+xy\cos x\quad\Rightarrow\dfrac{\partial M}{\partial y}=35x^{5}y^{4}+2\sin x+x\cos x$

$$N=6x^{6}y^{4}+2x\sin x\quad\Rightarrow\frac{\partial N}{\partial x}=36x^{5}y^{4}+2\sin x+2x\cos x$$

由判斷式 $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$ 可知此非正合微分方程

由 $\dfrac{1}{N}(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x})$ 與 $\dfrac{1}{M}(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y})$ 可知 μ 非單純為 x 函數或 y 函數

$\therefore$ 令 $\mu = x^m y^n$

同乘積分因子後可得

$$(7x^{m+5}y^{n+5} + 2x^m y^{n+1}\sin x + x^{m+1}y^{n+1}\cos x)dx + (6x^{m+6}y^{n+4} + 2x^{m+1}y^n \sin x)dy = 0$$

此為正合微分方程

$\therefore$ $\overline{M} = 7x^{m+5}y^{n+5} + 2x^m y^{n+1}\sin x + x^{m+1}y^{n+1}\cos x$

$$\Rightarrow \frac{\partial \overline{M}}{\partial y} = 7(n+5)x^{m+5}y^{n+4} + 2(n+1)x^m y^n \sin x + (n+1)x^{m+1}y^n \cos x$$

$\overline{N} = 6x^{m+6}y^{n+4} + 2x^{m+1}y^n \sin x$

$$\Rightarrow \frac{\partial \overline{N}}{\partial x} = 6(m+6)x^{m+5}y^{n+4} + 2(m+1)x^m y^n \sin x + 2x^{m+1}y^n \cos x$$

又 $\dfrac{\partial \overline{M}}{\partial y} = \dfrac{\partial \overline{N}}{\partial x}$ $\Rightarrow m = 1, n = 1$

$$\therefore \quad \overline{M} = \frac{\partial \phi}{\partial x} = 7x^6y^6 + 2xy^2\sin x + x^2y^2\cos x \quad \Rightarrow \phi = x^7y^6 + x^2y^2\sin x + f(y)$$

$$\overline{N} = \frac{\partial \phi}{\partial y} = 6x^7y^5 + 2x^2y\sin x \quad \Rightarrow \phi = x^7y^6 + x^2y^2\sin x + g(x)$$

由上兩式可知 $\phi(x, y) = x^7y^6 + x^2y^2\sin x = C$