國立臺灣海洋大學河海工程學系 2020 工程數學(一) 2A 班第一次作業參考解答

系級：________ 學號：________ 姓名：________

試求下述微分方程式:

1. (1) $y(1+x^2)y' = 3x + xy^2$

   (2) $e^{x+y}\dfrac{dy}{dx} = 2x$

   (3) $(x\ln x)y' = y$

   (4) $y' = x\csc^2 y$

2. $xy' = y^2 + y$

3. $y' = \dfrac{x^3 + y^3}{3xy^2}$

4. $(y^2 + 2y + 5)dx - dy = 0$

參考解答:

1. (1) $y(1+x^2)y' = 3x + xy^2 \quad \Rightarrow y(1+x^2)\dfrac{dy}{dx} = x(3+y^2)$

$$\Rightarrow \frac{y}{3+y^2}dy = \frac{x}{1+x^2}dx$$

(兩邊積分) $\Rightarrow \displaystyle\int \frac{y}{3+y^2}dy = \int \frac{x}{1+x^2}dx$

$$\Rightarrow \frac{1}{2}\ln(y^2+3) = \frac{1}{2}\ln(x^2+1) + C_1$$

$$\Rightarrow \ln(\frac{y^2+3}{x^2+1}) = C_2$$

$$\Rightarrow \frac{y^2+3}{x^2+1} = e^{C_2} = C$$

$$\Rightarrow y^2 = C(x^2+1) - 3$$

(2) $e^{x+y}\dfrac{dy}{dx} = 2x \quad \Rightarrow e^y dy = 2xe^{-x}dx$

(兩邊積分) $\Rightarrow \displaystyle\int e^y dy = \int 2xe^{-x}dx$

$$\Rightarrow e^y = -2(1+x)e^{-x} + C$$

(3) $(x\ln x)y' = y \quad \Rightarrow \dfrac{1}{y}dy = \dfrac{1}{x\ln x}dx$

(兩邊積分) $\Rightarrow \displaystyle\int \frac{1}{y}dy = \int \frac{1}{x\ln x}dx$

$$\Rightarrow \int \frac{1}{y} dy = \int \frac{1}{\ln x} d(\ln x)$$

$$\Rightarrow \ln y = \ln(\ln x) + \ln C$$

$$\Rightarrow y = C \ln x$$

(4) $y' = x\csc^2 y \quad \Rightarrow \frac{dy}{dx} = x\csc^2 y = \frac{x}{\sin^2 y}$

$$\Rightarrow \sin^2 y dy = x dx$$

(兩邊積分) $\Rightarrow \int \sin^2 y dy = \int x\, dx$

$$\Rightarrow \int \frac{1-\cos 2y}{2} dy = \frac{1}{2}x^2 + C_1$$

$$\Rightarrow \frac{1}{2}y - \frac{1}{4}\sin 2y = \frac{1}{2}x^2 + C_1$$

$$\Rightarrow 2y - \sin 2y = 2x^2 + C$$

2. $xy' = y^2 + y \quad \Rightarrow \frac{1}{y(y+1)} dy = \frac{1}{x} dx$

(兩邊積分) $\Rightarrow \int \frac{1}{y(y+1)} dy = \int \frac{1}{x} dx$

$$\Rightarrow \int \left(\frac{1}{y} - \frac{1}{y+1}\right) dy = \int \frac{1}{x} dx$$

$$\Rightarrow \ln|y| - \ln|y+1| = \ln|x| + \ln|C|$$

$$\Rightarrow \frac{y}{y+1} = C_1 x$$

$$\Rightarrow 1 + \frac{1}{y} = \frac{C}{x}$$

$$\Rightarrow \frac{1}{y} = \frac{C-x}{x}$$

$$\Rightarrow y = \frac{x}{C-x}$$

3. $y' = \frac{x^3 + y^3}{3xy^2} \quad \rightarrow$ 齊次型 ODE

$$\Rightarrow y' = \frac{1}{3}\left(\frac{x^2}{y^2} + \frac{y}{x}\right)$$

令 $u = \frac{y}{x} \quad \Rightarrow y = ux \quad \Rightarrow y' = u'x + u$

$$\therefore\quad y' = \frac{1}{3}\left(\frac{x^2}{y^2} + \frac{y}{x}\right) \quad \Rightarrow u'x + u = \frac{1}{3}\left(\frac{1}{u^2} + u\right)$$

$$\Rightarrow u'x = \frac{1-2u^3}{3u^2}$$

$$\Rightarrow \frac{3u^2}{1-2u^3}du = \frac{1}{x}dx$$

(兩邊積分) $$\Rightarrow \int \frac{3u^2}{1-2u^3}du = \int \frac{1}{x}dx$$

$$\Rightarrow -\frac{1}{2}\int \frac{1}{1-2u^3}d(1-2u^3) = \int \frac{1}{x}dx$$

$$\Rightarrow -\frac{1}{2}\ln\left|1-2u^3\right| = \ln|x| + \ln|C_1|$$

$$\Rightarrow \ln\left|1-2u^3\right| + 2\ln|x| = \ln|C|$$

$$\Rightarrow (1-2u^3)\cdot x^2 = C$$

$$\Rightarrow x^3 - 2y^3 = Cx$$

4. $(y^2+2y+5)dx - dy = 0 \Rightarrow \dfrac{1}{y^2+2y+5}dy = dx$

(兩邊積分) $$\Rightarrow \int \frac{1}{y^2+2y+5}dy = \int dx$$

$$\Rightarrow \int \frac{1}{(y+1)^2+4}dy = \int dx$$

令 $u = y+1 \quad \Rightarrow du = dy$

$$\int \frac{1}{(y+1)^2+4}dy = \int dx \quad \Rightarrow \int \frac{1}{u^2+4}du = \int dx$$

令 $u = 2\tan\theta \quad \Rightarrow du = 2\sec^2\theta\, d\theta$

$$\Rightarrow \int \frac{1}{u^2+4}du = \int dx \quad \Rightarrow \frac{1}{2}\int \frac{\sec^2\theta}{\tan^2\theta+1}d\theta = \int dx$$

$$\Rightarrow \frac{1}{2}\int d\theta = \int dx$$

$$\Rightarrow \theta = 2x + C$$

$$\Rightarrow \tan^{-1}\frac{u}{2} = 2x + C$$

$$\Rightarrow y+1 = 2\tan(2x+C)$$