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試求下述微分方程式：

1. $2yy' = e^{x-y^2}$, $y(4) = -2$
2. $x\sin(y) \cdot y' = \cos(y)$
3. $y' = 3x^2(y+2)$, $y(2) = 8$
4. $y' = 1 + 4y^2$, $y(1) = 0$
5. $yy' = 2x \cdot \sec(3y)$, $y(\frac{2}{3}) = \frac{\pi}{3}$
6. $x^2y' = x^2 + y^2$
7. $(x-2y)y' = 2x-y$
8. $y' = \frac{x+2y+7}{-2x+y-9}$

參考解答：

1. $\Rightarrow y = \pm\sqrt{x}$

由 $y(4) = -2$ 可知 $\Rightarrow y = -\sqrt{x}$

2. $x\sin(y) \cdot y' = \cos(y) \Rightarrow \frac{\sin(y)}{\cos(y)} y' = \frac{1}{x}$
 $\Rightarrow \int \frac{\sin(y)}{\cos(y)} dy = \int \frac{1}{x} dx$

$\Rightarrow -\ln|\cos(y)| = \ln|x| + \ln c$

若 $\cos(y) \neq 0$ 且 $x \neq 0$ 則 $\frac{1}{\cos(y)} = cx \Rightarrow \sec(y) = cx$

若 $\cos(y) = 0 \Rightarrow y = \frac{(2n+1)\pi}{2}$ 也滿足微分方程

3. $y' = 3x^2(y+2) \Rightarrow \int \frac{1}{y+2} dy = \int 3x^2 dx$
 $\Rightarrow \ln(y+2) = x^3 + c$

又 $y(2) = 8 \Rightarrow c = \ln 10 - 8$

$$\therefore \ln\left(\frac{y+2}{10}\right) = x^3 - 8 \Rightarrow y = 10e^{x^3-8} - 2$$

$$4. 2e^{y^2} = e^x + 2e^4 - e$$

$$5. yy' = 2x \cdot \sec(3y) \Rightarrow y \cdot \cos(3y)y' = 2x$$

$$\Rightarrow \int y \cos(3y) dy = \int 2x dx$$

$$\Rightarrow \frac{1}{3} y \sin(3y) + \frac{1}{9} \cos(3y) = x^2 + c$$

$$\text{又 } y\left(\frac{2}{3}\right) = \frac{\pi}{3} \Rightarrow \frac{\pi}{9} \sin(\pi) + \frac{1}{9} \cos(\pi) = \frac{4}{9} + c \Rightarrow c = -\frac{5}{9}$$

$$\therefore \frac{1}{3} y \sin(3y) + \frac{1}{9} \cos(3y) = x^2 - \frac{5}{9}$$

$$6. x^2 y' = x^2 + y^2 \Rightarrow y' = 1 + \frac{y^2}{x^2}$$

$$\text{令 } u = \frac{y}{x} \Rightarrow y = ux \Rightarrow \frac{dy}{dx} = \frac{du}{dx} x + u$$

$$\therefore \text{原式} \Rightarrow \frac{du}{dx} x + u = 1 + u^2 \Rightarrow \int \frac{1}{u^2 - u + 1} du = \int \frac{1}{x} dx$$

$$\Rightarrow \int \frac{1}{\left(u - \frac{1}{2}\right)^2 + \frac{3}{4}} du = \int \frac{1}{x} dx$$

$$\left(\text{令 } v = u - \frac{1}{2}\right) \Rightarrow \int \frac{1}{v^2 + \frac{3}{4}} dv = \int \frac{1}{x} dx$$

$$\left(\text{令 } v = \frac{\sqrt{3}}{2} w\right) \Rightarrow \frac{2\sqrt{3}}{3} \int \frac{1}{w^2 + 1} dw = \int \frac{1}{x} dx$$

$$\Rightarrow \frac{2\sqrt{3}}{3} \tan^{-1} w = \ln|x| + c$$

$$\Rightarrow \frac{2\sqrt{3}}{3} \tan^{-1} \frac{2y-x}{\sqrt{3}x} = \ln|x| + c$$

$$7. (x-2y)y' = 2x-y \Rightarrow y' = \frac{2x-y}{x-2y} \Rightarrow \frac{dy}{dx} = \frac{2 - \frac{y}{x}}{1 - 2\frac{y}{x}}$$

$$\text{令 } u = \frac{y}{x} \Rightarrow y = ux \Rightarrow \frac{dy}{dx} = \frac{du}{dx} x + u$$

$$\begin{aligned} \therefore \text{原式} &\Rightarrow \frac{du}{dx}x+u = \frac{2-u}{1-2u} \Rightarrow \frac{du}{dx}x = \frac{2-2u+2u^2}{1-2u} \\ &\Rightarrow \frac{1}{2} \int \frac{1-2u}{1-u+u^2} du = \int \frac{1}{x} dx \end{aligned}$$

$$\text{令 } v=1-u+u^2 \Rightarrow dv=(-1+2u)du$$

$$\begin{aligned} \therefore \text{原式} &\Rightarrow -\frac{1}{2} \int \frac{1}{v} dv = \int \frac{1}{x} dx \Rightarrow -\frac{1}{2} \ln v = \ln x + \ln c \\ &\Rightarrow v^{-\frac{1}{2}} = cx \\ &\Rightarrow x^2 - xy + y^2 = c_1 \end{aligned}$$

$$8. y' = \frac{x+2y+7}{-2x+y-9}$$

$$\text{令 } x=X+a, y=Y+b$$

$$\text{原式} \Rightarrow \frac{dY}{dX} = \frac{X+2Y+(a+2b+7)}{-2X+Y+(-2a+b-9)}$$

$$\Rightarrow \begin{cases} a+2b+7=0 \\ -2a+b-9=0 \end{cases} \Rightarrow \begin{cases} a=-5 \\ b=-1 \end{cases}$$

$$\therefore \text{原式} \Rightarrow \frac{dY}{dX} = \frac{X+2Y}{-2X+Y} \Rightarrow \frac{dY}{dX} = \frac{1+\frac{Y}{X}}{-2+\frac{Y}{X}}$$

$$\text{令 } u = \frac{Y}{X} \Rightarrow Y = uX \Rightarrow \frac{dY}{dX} = \frac{du}{dX}X + u$$

$$\begin{aligned} \therefore \text{原式} &\Rightarrow \frac{du}{dX}X + u = \frac{1+2u}{-2+u} \Rightarrow \frac{du}{dX}X + u = \frac{1+2u}{-2+u} - u \\ &\Rightarrow \int \frac{-2+u}{1+4u-u^2} du = \int \frac{1}{X} dX \end{aligned}$$

$$\text{令 } v=1+4u-u^2 \Rightarrow dv=(4-2u)du$$

$$\begin{aligned} \therefore \text{原式} &\Rightarrow -\frac{1}{2} \int \frac{1}{v} dv = \int \frac{1}{X} dX \Rightarrow -\frac{1}{2} \ln v = \ln X + \ln c \\ &\Rightarrow v^{-\frac{1}{2}} = cX \\ &\Rightarrow (x+5)^2 + 4(x+5)(y+1) - (y+1)^2 = c_1 \end{aligned}$$