國立臺灣海洋大學河海工程學系 2019 工程數學(一) 期末考參考解答

系級：________ 學號：________ 姓名：________

1. 已知矩陣 $A=\begin{bmatrix}3&2&0&0\\4&3&0&0\\0&0&6&5\\0&0&7&6\end{bmatrix}$

   (1) 試求 A 的行列式值。 (6%)

   (2) 試以 Guass-Jordan 消去法求 A^{-1}。 (10%)

2. 給一矩陣 $A=\begin{bmatrix}9&7&a\\-4&2&b\\7&11&c\end{bmatrix}$，其中 a、b、c 均為常數，並且知道矩陣 A 有兩特徵向量為 $x^1=\begin{Bmatrix}1\\0\\1\end{Bmatrix}$ 與 $x^2=\begin{Bmatrix}1\\1\\2\end{Bmatrix}$，試問:

   (1) a、b、c 之值為何? (6%)

   (2)矩陣 A 之特徵值為何? (6%)

   (3)第三個特徵向量 $x^3=?$ (4%)

3. 已知 $A=\begin{bmatrix}4&3\\6&3\end{bmatrix}$，$B^{-1}=\dfrac{1}{4}\begin{bmatrix}5&2\\3&2\end{bmatrix}$，試求:

   (1) $\det(5A^T)=?$ (2) $\det(B^T)=?$ (3) $\det(AB)=?$ (4) $(AB)^{-1}=?$ (16%)

4. 試以 Gram-Schmidt 法將向量集$\{x^1, x^2, x^3\}$, $x^1=[1\ 1\ 1]^T$, $x^2=[2\ 0\ 1]^T$, $x^3=[1\ 2\ 2]^T$ 正交單位化。 (10%)

5. (1) 給方程式 $3x_1^2-2x_1x_2+3x_2^2=16$，試問此二次式代表何種圓錐曲線? (10%)
   (請將之轉換至主軸，即將舊座標向量 $\mathbf{x}^T=[x_1\ x_2]$ 轉換至新座標向量 $\mathbf{y}^T=[y_1\ y_2]$)

   (2) 請問在曲線某一點，其距離原點最遠的距離為何? (2%)

6. 已知 $A=\begin{bmatrix}1&2&1\\0&-1&1\\0&-6&4\end{bmatrix}$，試問:

   (1) A 的特徵方程為何? (4%)

   (2) $A^{10}-2A^9-3A^2-3I=?$ (8%)

   (3) 若 $A^{-1}=pA^2+qA+rI$ ，則 $p=?$, $q=?$, $r=?$ (8%)

7. 試解: $\dfrac{dx}{dt}=Ax+z$ 其中 $A=\begin{bmatrix}3&2\\4&1\end{bmatrix}$, $z=\begin{Bmatrix}e^{-t}\\4e^{-t}\end{Bmatrix}$。 (10%)

參考解答:

1. 已知矩陣 $A=\begin{bmatrix}3&2&0&0\\4&3&0&0\\0&0&6&5\\0&0&7&6\end{bmatrix}$

(1) 試求 A 的行列式值。 (6%)

(2) 試以 Guass-Jordan 消去法求 A^{-1} 。 (10%)

(1) $\det(A)=\begin{vmatrix}3&2&0&0\\4&3&0&0\\0&0&6&5\\0&0&7&6\end{vmatrix}=\begin{vmatrix}3&2\\4&3\end{vmatrix}\cdot\begin{vmatrix}6&5\\7&6\end{vmatrix}=1\cdot1=1$

(2) $\left[\begin{array}{cccc|cccc}3&2&0&0&1&0&0&0\\4&3&0&0&0&1&0&0\\0&0&6&5&0&0&1&0\\0&0&7&6&0&0&0&1\end{array}\right]\xrightarrow[R_{34}(-\frac{7}{6})]{R_{12}(-\frac{4}{3})}\left[\begin{array}{cccc|cccc}3&2&0&0&1&0&0&0\\0&\frac{1}{3}&0&0&-\frac{4}{3}&1&0&0\\0&0&6&5&0&0&1&0\\0&0&0&\frac{1}{6}&0&0&-\frac{7}{6}&1\end{array}\right]$

$\xrightarrow[R_4(6)]{R_2(3)}\left[\begin{array}{cccc|cccc}3&2&0&0&1&0&0&0\\0&1&0&0&-4&3&0&0\\0&0&6&5&0&0&1&0\\0&0&0&1&0&0&-7&6\end{array}\right]$

$\xrightarrow[R_{43}(-5)]{R_{21}(-2)}\left[\begin{array}{cccc|cccc}3&0&0&0&9&-6&0&0\\0&1&0&0&-4&3&0&0\\0&0&6&0&0&0&36&-30\\0&0&0&1&0&0&-7&6\end{array}\right]$

$\xrightarrow[R_{43}(-5)]{R_{21}(-2)}\left[\begin{array}{cccc|cccc}1&0&0&0&3&-2&0&0\\0&1&0&0&-4&3&0&0\\0&0&1&0&0&0&6&-5\\0&0&0&1&0&0&-7&6\end{array}\right]$

2. 給一矩陣 $A=\begin{bmatrix}9&7&a\\-4&2&b\\7&11&c\end{bmatrix}$，其中 a、b、c 均為常數，並且知道矩陣 A 有兩特

徵向量為 $x^1=\begin{Bmatrix}1\\0\\1\end{Bmatrix}$ 與 $x^2=\begin{Bmatrix}1\\1\\2\end{Bmatrix}$，試問:

(1) a、b、c 之值為何? (6%)
(2)矩陣 A 之特徵值為何? (6%)
(3)第三個特徵向量 $x^3 = ?$ (4%)

(1) 由特徵值的定義 $Ax = \lambda x$ 可知

$$\begin{bmatrix} 9 & 7 & a \\ -4 & 2 & b \\ 7 & 11 & c \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} = \lambda_1 \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} \quad \Rightarrow \begin{cases} 9 + a = \lambda_1 \\ -4 + b = 0 \\ 7 + c = \lambda_1 \end{cases} \quad \Rightarrow b = 4$$

$$\begin{bmatrix} 9 & 7 & a \\ -4 & 2 & b \\ 7 & 11 & c \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 2 \end{Bmatrix} = \lambda_2 \begin{Bmatrix} 1 \\ 1 \\ 2 \end{Bmatrix} \quad \Rightarrow \begin{cases} 9 + 7 + 2a = \lambda_2 \\ -4 + 2 + 2b = \lambda_2 \\ 7 + 11 + 2c = 2\lambda_2 \end{cases} \begin{matrix} \lambda_2 = 6 \\ \Rightarrow a = -5 \\ c = -3 \end{matrix}$$

$\therefore$ 可得 $\lambda_1 = 4$

(2) $trace(A) = \lambda_1 + \lambda_2 + \lambda_3 \quad \Rightarrow 9 + 2 - 3 = 4 + 6 + \lambda_3 \quad \Rightarrow \lambda_3 = -2$
$\therefore \lambda_1 = 4$, $\lambda_2 = 6$, $\lambda_3 = -2$

(3) $$Ax^3 = \lambda x^3 \quad \Rightarrow (A - \lambda I)x^3 = 0 \quad \Rightarrow \begin{bmatrix} 11 & 7 & -5 \\ -4 & 4 & 4 \\ 7 & 11 & -1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \begin{cases} 11x_1 + 7x_2 - 5x_3 = 0 \\ -4x_1 + 4x_2 + 4x_3 = 0 \end{cases} \quad \Rightarrow x_1 = t,\ x_2 = -\frac{t}{2},\ x_3 = \frac{3t}{2}$$

$$\therefore \quad x^3 = \begin{Bmatrix} 1 \\ -\frac{1}{2} \\ \frac{3}{2} \end{Bmatrix} t \quad 取 \quad t = 2 \quad \Rightarrow x^3 = \begin{Bmatrix} 2 \\ -1 \\ 3 \end{Bmatrix}$$

3. 已知 $A = \begin{bmatrix} 4 & 3 \\ 6 & 3 \end{bmatrix}$，$B^{-1} = \frac{1}{4}\begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix}$，試求:
(1) $\det(5A^T) = ?$ (2) $\det(B^T) = ?$ (3) $\det(AB) = ?$ (4) $(AB)^{-1} = ?$ (16%)

(1) $\det(A) = \begin{vmatrix} 4 & 3 \\ 6 & 3 \end{vmatrix} = -6$

$\det(5A^T) = \det(5A) = 5^2 \cdot \det(A^T) = 25 \cdot (-6) = -150$

(2) $\det(B^{-1}) = \begin{vmatrix} \frac{5}{4} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{2} \end{vmatrix} = \frac{1}{4}$

$BB^{-1} = I \quad \Rightarrow \det(BB^{-1}) = \det(I) \quad \Rightarrow \det(B)\cdot\det(B^{-1}) = 1 \quad \Rightarrow \det(B) = 4$

$\det(B^T) = \det(B) = 4$

(3) $\det(AB) = \det(A)\cdot\det(B) = -24$

(4) $A = \begin{bmatrix} 4 & 3 \\ 6 & 3 \end{bmatrix} \quad \Rightarrow A^{-1} = -\frac{1}{6}\begin{bmatrix} 3 & -3 \\ -6 & 4 \end{bmatrix}$

$$(AB)^{-1} = B^{-1}A^{-1} = -\frac{1}{24}\begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 3 & -3 \\ -6 & 4 \end{bmatrix}$$

$$= -\frac{1}{24}\begin{bmatrix} 3 & -7 \\ -3 & -1 \end{bmatrix} = \frac{1}{24}\begin{bmatrix} -3 & 7 \\ 3 & 1 \end{bmatrix}$$

4. 試以 Gram-Schmidt 法將向量集 $\{x^1, x^2, x^3\}$, $x^1 = [1\ \ 1\ \ 1]^T$, $x^2 = [2\ \ 0\ \ 1]^T$, $x^3 = [1\ \ 2\ \ 2]^T$ 正交單位化。 (10%)

$x^1 = [1\ \ 1\ \ 1]^T \quad \Rightarrow u^1 = \frac{x^1}{\|x^1\|} = [\frac{1}{\sqrt{3}}\ \ \frac{1}{\sqrt{3}}\ \ \frac{1}{\sqrt{3}}]^T$

$b^2 = x^2 - < x^2, u^1 > u^1 = [1\ \ -1\ \ 0]^T$

$\Rightarrow u^2 = \frac{b^2}{\|b^2\|} = [\frac{1}{\sqrt{2}}\ \ -\frac{1}{\sqrt{2}}\ \ 0]^T$

$b^3 = x^3 - < x^3, u^1 > u^1 - < x^3, u^2 > u^2 = [-\frac{1}{6}\ \ -\frac{1}{6}\ \ \frac{1}{3}]^T$

$\Rightarrow u^3 = \frac{b^3}{\|b^3\|} = [-\frac{1}{\sqrt{6}}\ \ -\frac{1}{\sqrt{6}}\ \ \frac{2}{\sqrt{6}}]^T$

5. (1) 給方程式 $3x_1^2 - 2x_1x_2 + 3x_2^2 = 16$，試問此二次式代表何種圓錐曲線? (10%)
(請將之轉換至主軸，即將舊座標向量 $\mathbf{x}^T = [x_1\ x_2]$ 轉換至新座標向量 $\mathbf{y}^T = [y_1\ y_2]$)

(2) 請問在曲線某一點，其距離原點最遠的距離為何? (2%)

(1) $3x_1^2 - 2x_1x_2 + 3x_2^2 = 16 \quad \Rightarrow \mathbf{x}^T\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}\mathbf{x} = 16$ 其中 $\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$

由 $|A - \lambda I| = 0 \quad \Rightarrow \begin{vmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix} = 0 \quad \Rightarrow \lambda = 2$ or 4

當 $\lambda=2$ 時，$\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix} \Rightarrow x^1=\begin{Bmatrix}x_1\\ x_2\end{Bmatrix}=\frac{1}{\sqrt{2}}\begin{Bmatrix}1\\ 1\end{Bmatrix}$

當 $\lambda=4$ 時，$\begin{bmatrix}-1 & -1\\ -1 & -1\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix} \Rightarrow x^1=\begin{Bmatrix}x_1\\ x_2\end{Bmatrix}=\frac{1}{\sqrt{2}}\begin{Bmatrix}-1\\ 1\end{Bmatrix}$

$\therefore\ s=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix} \quad \Rightarrow s^{-1}=s^T=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$

令 $y=s^T x$

$$x^T\begin{bmatrix}3 & -1\\ -1 & 3\end{bmatrix}x=16 \quad \Rightarrow x^T s\begin{bmatrix}2 & 0\\ 0 & 4\end{bmatrix}s^T x=16$$

$$\Rightarrow y^T\begin{bmatrix}2 & 0\\ 0 & 4\end{bmatrix}y=16$$

$$\Rightarrow 2y_1^2+4y_2^2=16$$

$$\Rightarrow \frac{y_1^2}{(2\sqrt{2})^2}+\frac{y_2^2}{2^2}=1$$

$\therefore$ 此為橢圓曲線

(2) 離原點最遠距離為 $2\sqrt{2}$

6. 已知 $A=\begin{bmatrix}1 & 2 & 1\\ 0 & -1 & 1\\ 0 & -6 & 4\end{bmatrix}$，試問:

(1) A 的特徵方程為何? (4%)

(2) $A^{10}-2A^9-3A^2-3I=$? (8%)

(3) 若 $A^{-1}=pA^2+qA+rI$ ，則 $p=?,\ q=?,\ r=?$ (8%)

(1) $|A-\lambda I|=0 \quad \Rightarrow \begin{vmatrix}1-\lambda & 2 & 1\\ 0 & -1-\lambda & 1\\ 0 & -6 & 4-\lambda\end{vmatrix}=0$

$$\Rightarrow \lambda^3-4\lambda^2+5\lambda-2=0$$
$$\Rightarrow (\lambda-1)^2(\lambda-2)=0$$
$$\Rightarrow \lambda=1,1,2$$

(2) 由 Cayley-Hamilton 定理可知: $A^3-4A^2+5A-2I=0$

$A^{10}-2A^9-3A^2-3I=Q(A)\cdot(A^3-4A^2+5A-2I)+pA^2+qA+rI$

$\Rightarrow \lambda^{10}-2\lambda^9-3\lambda^2-3=Q(\lambda)\cdot(\lambda^3-4\lambda^2+5\lambda-2)+p\lambda^2+q\lambda+r$

代入 $\lambda=2$ 可得 $-15=4p+2q+r \quad \ldots (1)$

代入 $\lambda = 1$ 可得 $-7 = p + q + r \quad \ldots (2)$

微分後代入 $\lambda = 1$ 可得 $-14 = 2p + q \quad \ldots (3)$

解聯立後可得 $p = 6$，$q = -26$，$r = 13$

$$\therefore A^{10} - 2A^9 - 3A^2 - 3I = 6A^2 - 26A + 13I = \begin{bmatrix} -7 & -88 & 16 \\ 0 & 9 & -8 \\ 0 & 48 & -31 \end{bmatrix}$$

(3) $A^3 - 4A^2 + 5A - 2I = 0 \quad \Rightarrow A^{-1}(A^3 - 4A^2 + 5A - 2I) = 0$

$$\Rightarrow A^{-1} = \frac{1}{2}A^2 - 2A + \frac{5}{2}I = \begin{bmatrix} 1 & -7 & \frac{3}{2} \\ 0 & 2 & -\frac{1}{2} \\ 0 & 3 & -\frac{1}{2} \end{bmatrix}$$

$p = \frac{1}{2}$，$q = -2$，$r = \frac{5}{2}$

7. 試解: $\frac{dx}{dt} = Ax + z$ 其中 $A = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$, $z = \begin{Bmatrix} e^{-t} \\ 4e^{-t} \end{Bmatrix}$。 (10%)

$$\det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} 3-\lambda & 2 \\ 4 & 1-\lambda \end{vmatrix} = (\lambda - 5)(\lambda + 1) = 0 \Rightarrow \lambda = 5 \text{ or } -1$$

當 $\lambda = 5 \quad \Rightarrow \begin{bmatrix} -2 & 2 \\ 4 & -4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \Rightarrow x^2 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

當 $\lambda = -1 \quad \Rightarrow \begin{bmatrix} 4 & 2 \\ 4 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \Rightarrow x^1 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -2 \end{Bmatrix}$

$$\therefore \quad A = SDS^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}^{-1}$$

令 $x = Sy \quad \Rightarrow S\frac{dy}{dt} = ASy + z \quad \Rightarrow \frac{dy}{dt} = S^{-1}ASy + S^{-1}z$

$$\therefore \quad \begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{Bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}^{-1} \begin{Bmatrix} e^{-t} \\ 4e^{-t} \end{Bmatrix}$$

$$\Rightarrow \begin{cases} \dot{y}_1 = 5y_1 + 2e^{-t} \\ \dot{y}_2 = -y_2 - e^{-t} \end{cases} \quad \Rightarrow \begin{cases} y_1 = c_1 e^{5t} - \frac{1}{3}e^{-t} \\ y_2 = c_2 e^{-t} - te^{-t} \end{cases}$$

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} y_1 + y_2 \\ y_1 - 2y_2 \end{Bmatrix} = \begin{Bmatrix} c_1 e^{5t} + c_2 e^{-t} - \frac{1}{3} e^{-t} - t e^{-t} \\ c_1 e^{5t} - 2c_2 e^{-t} - \frac{1}{3} e^{-t} + 2t e^{-t} \end{Bmatrix}$$