

系級：_____ 學號：_____ 姓名：_____

1. 三角形之三頂點為 $A(1, 2, 3)$, $B(2, -1, 3)$, $C(-2, 3, 2)$, 試問其三內角為何? (9%)
(106 交大土木丁組)

2. (1) 向量場 $\vec{F} = 2xy\vec{i} + (x^2 + 2yz)\vec{j} + (y^2 + 1)\vec{k}$, 試說明 \vec{F} 是否為保守場? (6%)

(2) 若給一純量函數 $\phi(x, y, z)$ 並且滿足 $\vec{F} = -\nabla\phi$, 試求 $\phi(x, y, z)$ 為何? (6%)

(3) 試計算 $\int_C \vec{F} \cdot d\vec{r}$, 其中 $C: \cos t\vec{i} + \sin t\vec{j} + 2t\vec{k}$, $0 \leq t \leq \frac{\pi}{2}$ (8%)

(103 成大土木甲乙丁組)

3. 給一心臟線 $\vec{r}(t) = (\rho(t)\cos t, \rho(t)\sin t)$, 其中 $\rho(t) = a(1 - \cos t)$ 且 $0 \leq t \leq 2\pi$

(1) 試計算心臟線之周長。 (7%)

(2) 試以格林定理: $\oint f dx + g dy = \iint (\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}) dx dy$, 紿定 $f = -y$ 與 $g = x$ 來求
心臟線之面積。 (7%)

(3) 試求在點 $(0, a)$ 之曲率 κ 。 (7%)

4. 曲線 C 為空間上兩個面 S_1 和 S_2 的交線, S_1 的方程為 $x^2 + y^2 + z^2 = 1$, S_2 的方
程為 $y = z$, S_1 為球面, S_2 為平面。曲線 C 上兩點 A 和 B 的座標分別是 $(1, 0, 0)$

和 $(\frac{\sqrt{2}}{2}, \frac{1}{2}, \frac{1}{2})$, 請計算沿著曲線 C 由 A 點積分到 B 點的路徑積分

$\int_{(1, 0, 0)}^{(\frac{\sqrt{2}}{2}, \frac{1}{2}, \frac{1}{2})} u dx + v dy + w dz$, 其中 $u = y^2 z^2 - y \sin(xy)$, $v = 2xyz^2 - x \sin(xy)$,

$w = 2xy^2 z$ (15%)

(106 中央土木結構組)

5. 已知場 $\vec{F} = z^2 e^{x^2} \vec{i} + xy^2 \vec{j} + \tan^{-1} y \vec{k}$ 及曲面 $S_1: x^2 + y^2 = 4 - z$ ($0 \leq z \leq 4$)

與 $S_2: z = 0$, Γ 為 S_1 、 S_2 之交線, 試問:

(1) 試畫出 S_1 之圖形, 並標出 S_1 、 S_2 與 Γ 。 (5%)

(2) \vec{F} 是否為保守場? 請說明之。 (5%)

(3) S_1 上的單位法向量 $\vec{n} = ?$ (5%)

(4) $\iint_S (\nabla \times \vec{F}) \cdot \vec{n} dS = ?$ (5%)

(5) $\int_{\Gamma} \vec{F} \cdot d\vec{r} = ?$ (5%)

(6) $\iint_{S_1} (\nabla \times \vec{F}) \cdot \vec{n} dA = ?$ (5%)

(7) $\iint_{S_2} (\nabla \times \vec{F}) \cdot \vec{n} dA = ?$ (5%)

Hint:

Gauss 散度定理: $\iiint \nabla \cdot \vec{F} dV = \iint \vec{F} \cdot \vec{n} dA$ (3D)

$\iint \nabla \cdot \vec{F} dA = \oint \vec{F} \cdot \vec{n} ds$ (2D)

格林定理: $\int P dx + Q dy = \iint (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dx dy$

Stokes 旋度定理: $\iint (\nabla \times \vec{F}) \cdot \vec{n} dA = \oint \vec{F} \cdot d\vec{r}$

曲率: $\kappa = \frac{|y''(x)|}{[1 + (y'(x))^2]^{\frac{3}{2}}} = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{[\vec{r}'(t) \cdot \vec{r}'(t)]^{\frac{3}{2}}}$ **扭率:** $\tau = \left| \frac{d(\vec{T}(s) \times \vec{N}(s))}{ds} \right|$

二倍角公式: $\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$, $\sin^2 \theta = \frac{1 - \cos 2\theta}{2}$

四面體體積: $V = \frac{1}{3} A_0 h$, (A_0 : 底面積; h : 高)

參考解答：

1. 三頂點為 $A(1,2,3)$, $B(2,-1,3)$, $C(-2,3,2)$

可知 $\overrightarrow{AB} = (1, -3, 0)$, $\overrightarrow{AC} = (-3, 1, -1)$

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = |\overrightarrow{AB}| |\overrightarrow{AC}| \cos \theta_A$$

$$\Rightarrow \theta_A = \cos^{-1} \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{|\overrightarrow{AB}| |\overrightarrow{AC}|} = \cos^{-1} \frac{-6}{\sqrt{10} \cdot \sqrt{11}} = 2.1798 \text{ (rad)} = 124.90^\circ$$

由 $\overrightarrow{BA} = (-1, 3, 0)$, $\overrightarrow{BC} = (-4, 4, -1)$

$$\overrightarrow{BA} \cdot \overrightarrow{BC} = |\overrightarrow{BA}| |\overrightarrow{BC}| \cos \theta_B$$

$$\Rightarrow \theta_B = \cos^{-1} \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}| |\overrightarrow{BC}|} = \cos^{-1} \frac{16}{\sqrt{10} \cdot \sqrt{33}} = 0.4933 \text{ (rad)} = 28.26^\circ$$

由 $\overrightarrow{CA} = (3, -1, 1)$, $\overrightarrow{CB} = (4, -4, 1)$

$$\overrightarrow{CA} \cdot \overrightarrow{CB} = |\overrightarrow{CA}| |\overrightarrow{CB}| \cos \theta_C$$

$$\Rightarrow \theta_C = \cos^{-1} \frac{\overrightarrow{CA} \cdot \overrightarrow{CB}}{|\overrightarrow{CA}| |\overrightarrow{CB}|} = \cos^{-1} \frac{17}{\sqrt{11} \cdot \sqrt{33}} = 0.4685 \text{ (rad)} = 26.84^\circ$$

$$2. (1) \because \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & x^2 + 2yz & y^2 + 1 \end{vmatrix} = 0$$

\therefore 可知 \vec{F} 為保守場

$$(2) \nabla \phi = -\vec{F} = \frac{\partial \phi}{\partial x} \vec{i} + \frac{\partial \phi}{\partial y} \vec{j} + \frac{\partial \phi}{\partial z} \vec{k}$$

$$\Rightarrow \begin{cases} \frac{\partial \phi}{\partial x} = -2xy \\ \frac{\partial \phi}{\partial y} = -x^2 - 2yz \\ \frac{\partial \phi}{\partial z} = -y^2 - 1 \end{cases} \Rightarrow \begin{cases} \phi = -x^2 y + f_1(y, z) \\ \phi = -x^2 y - y^2 z + f_2(x, z) \\ \phi = -y^2 z - z + f_3(x, y) \end{cases}$$

\therefore 比較後可得 $\phi(x, y, z) = -x^2 y - y^2 z - z + c$

$$(3) \ C : \cos t \vec{i} + \sin t \vec{j} + 2t \vec{k}, \ t : 0 \rightarrow \frac{\pi}{2}$$

可知 $(x, y, z) = (1, 0, 0) \rightarrow (0, 1, \pi)$

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \int_{(x_1, y_1, z_1)}^{(x_2, y_2, z_2)} -\nabla \phi \cdot d\vec{r} = - \int_{(x_1, y_1, z_1)}^{(x_2, y_2, z_2)} d\phi = \phi(x_1, y_1, z_1) - \phi(x_2, y_2, z_2) \\ &= \phi(1, 0, 0) - \phi(0, 1, \pi) = 2\pi \end{aligned}$$

$$3. (1) \ \vec{r} = x \vec{i} + y \vec{j} = \rho(t) \cos t \vec{i} + \rho(t) \sin t \vec{j}$$

$$\Rightarrow \begin{cases} x = a(1 - \cos t) \cos t \\ y = a(1 - \cos t) \sin t \end{cases} \Rightarrow \begin{cases} \frac{dx}{dt} = a(-\sin t + \sin 2t) \\ \frac{dy}{dt} = a(\cos t - \cos 2t) \end{cases}$$

$$S = \int ds = \int |d\vec{r}| = \int \sqrt{dx^2 + dy^2} = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$= 2a \int_0^\pi \sqrt{(-\sin t + \sin 2t)^2 + (\cos t - \cos 2t)^2} dt$$

$$= 2\sqrt{2} a \int_0^\pi \sqrt{1 - \sin t \cdot \sin 2t - \cos t \cdot \cos 2t} dt$$

$$= 2\sqrt{2} a \int_0^\pi \sqrt{1 - 2\sin^2 t \cdot \cos t - \cos^3 t + \sin^2 t \cdot \cos t} dt$$

$$= 2\sqrt{2} a \int_0^\pi \sqrt{1 - \cos t} dt$$

$$= 4a \int_0^\pi \sin \frac{t}{2} dt$$

$$= -8a \cdot \cos \frac{t}{2} \Big|_0^\pi$$

$$= 8a$$

$$(2) \oint -ydx + xdy = 2 \iint dxdy = 2A$$

$$\Rightarrow A = \frac{1}{2} \oint -ydx + xdy$$

$$= a^2 \int_0^\pi (1 - \cos t)(\sin^2 t - \sin t \cdot \sin 2t + \cos^2 t - \cos t \cdot \cos 2t) dt$$

$$= a^2 \int_0^\pi (1 - \cos t)^2 dt$$

$$= a^2 \int_0^\pi (1 - 2\cos t + \cos^2 t) dt$$

$$\begin{aligned}
&= a^2 \int_0^\pi (1 - 2\cos t + \frac{1 + \cos 2t}{2}) dt \\
&= a^2 \left(\frac{3}{2}t - 2\sin t + \frac{1}{4}\sin 2t \right) \Big|_0^\pi \\
&= \frac{3}{2}\pi a^2 \\
(3) \quad \kappa &= \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{[\vec{r}'(t) \cdot \vec{r}'(t)]^{\frac{3}{2}}} \\
\vec{r}'(t) &= x' \vec{i} + y' \vec{j} \\
\vec{r}''(t) &= x'' \vec{i} + y'' \vec{j} \\
\kappa &= \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{[(x')^2 + (y')^2]^{\frac{3}{2}}} = \frac{|x'y'' - x''y'|}{[(x')^2 + (y')^2]^{\frac{3}{2}}} \\
x = a(1 - \cos t) \cos t &\Rightarrow x' = \frac{dx}{dt} = a(-\sin t + \sin 2t) \\
&\Rightarrow x'' = \frac{d^2x}{dt^2} = a(-\cos t + 2\cos 2t) \\
y = a(1 - \cos t) \sin t &\Rightarrow y' = \frac{dy}{dt} = a(\cos t - \cos 2t) \\
&\Rightarrow y'' = \frac{d^2y}{dt^2} = a(-\sin t + 2\sin 2t) \\
\kappa &= \frac{|x'y'' - x''y'|}{[(x')^2 + (y')^2]^{\frac{3}{2}}} = \frac{3a^2(1 - \cos t)}{a^3[2(1 - \cos t)]^{\frac{3}{2}}} = \frac{3}{2\sqrt{2}a \cdot \sqrt{1 - \cos t}} \\
\text{將點 } (0, a) \text{ 即 } t = \frac{\pi}{2} \text{ 代入，可得 } \kappa &= \frac{3}{2\sqrt{2}a}
\end{aligned}$$

4. $\vec{F} = u \vec{i} + v \vec{j} + w \vec{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^2 - y\sin(xy) & 2xyz^2 - x\sin(xy) & 2xy^2z \end{vmatrix} = 0$$

\therefore 可知 \vec{F} 為保守場

$$(2) \quad \nabla \phi = \frac{\partial \phi}{\partial x} \vec{i} + \frac{\partial \phi}{\partial y} \vec{j} + \frac{\partial \phi}{\partial z} \vec{k} = \vec{F}$$

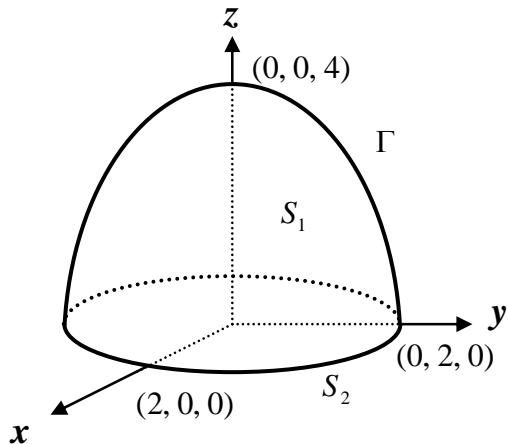
$$\Rightarrow \begin{cases} \frac{\partial \phi}{\partial x} = y^2 z^2 - y \sin(xy) \\ \frac{\partial \phi}{\partial y} = 2xyz^2 - x \sin(xy) \\ \frac{\partial \phi}{\partial z} = 2xy^2 z \end{cases} \Rightarrow \begin{cases} \phi = xy^2 z^2 + \cos(xy) + f_1(y, z) \\ \phi = xy^2 z^2 + \cos(xy) + f_2(x, z) \\ \phi = xy^2 z^2 + f_3(x, y) \end{cases}$$

\therefore 比較後可得 $\phi(x, y, z) = xy^2 z^2 + \cos(xy) + c$

$$\int_{(1,0,0)}^{(\frac{\sqrt{2}}{2}, \frac{1}{2}, \frac{1}{2})} u dx + v dy + w dz = \int_{(1,0,0)}^{(\frac{\sqrt{2}}{2}, \frac{1}{2}, \frac{1}{2})} \nabla \phi \cdot d\vec{r} = \int_{(1,0,0)}^{(\frac{\sqrt{2}}{2}, \frac{1}{2}, \frac{1}{2})} d\phi$$

$$= \phi\left(\frac{\sqrt{2}}{2}, \frac{1}{2}, \frac{1}{2}\right) - \phi(1, 0, 0) = \frac{\sqrt{2}}{32} + \cos \frac{\sqrt{2}}{4} - 1$$

5. (1)



$$(2) \quad \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 e^{x^2} & xy^2 & \tan^{-1} y \end{vmatrix} = \frac{1}{1+y^2} \vec{i} + 2ze^{x^2} \vec{j} + y^2 \vec{k}$$

$$\therefore \nabla \times \vec{F} \neq 0$$

$\therefore \vec{F}$ 不是保守場

$$(3) \quad \text{令 } \phi = x^2 + y^2 + z - 4$$

$$\vec{n} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{2x \vec{i} + 2y \vec{j} + \vec{k}}{\sqrt{4x^2 + 4y^2 + 1}}$$

$$(4) \quad \text{Gauss 散度定理: } \iint \vec{P} \cdot \vec{n} dS = \iiint \nabla \cdot \vec{P} dV$$

$$\text{令 } \vec{P} = \nabla \times \vec{F}$$

可知 $\nabla \cdot \vec{P} = \nabla \cdot (\nabla \times \vec{F}) = 0$ (\because 任何旋轉場不會有發散性)

因此 $\oint (\nabla \times \vec{F}) \cdot \vec{n} dS = 0$

(5) 由線積分 $\int_{\Gamma} \vec{F} \cdot d\vec{r} = \int_{\Gamma} z^2 e^{x^2} dx + xy^2 dy + \tan^{-1} y dz$

若令 $x = 2\cos\theta$, $y = 2\sin\theta$ 來求解可看出甚為複雜，求解非常不易

\therefore 可由 Stokes 定理: $\int_{\Gamma} \vec{F} \cdot d\vec{r} = \iint_{S_1} (\nabla \times \vec{F}) \cdot \vec{n} dA = - \iint_{S_2} (\nabla \times \vec{F}) \cdot \vec{n} dA$ 來求解

由 S_2 可知 $\vec{n} = -\vec{k}$

$$\therefore \int_{\Gamma} \vec{F} \cdot d\vec{r} = - \iint_{S_2} (\nabla \times \vec{F}) \cdot \vec{n} dA = \iint_{S_2} y^2 dA = \int_0^{2\pi} \int_0^2 (r \sin \theta)^2 r dr d\theta = 4\pi$$

(6) $\iint_{S_1} (\nabla \times \vec{F}) \cdot \vec{n} dA = \int_{\Gamma} \vec{F} \cdot d\vec{r} = 4\pi$

(7) $\iint_{S_2} (\nabla \times \vec{F}) \cdot \vec{n} dA = - \int_{\Gamma} \vec{F} \cdot d\vec{r} = -4\pi$