

系級：_____ 學號：_____ 姓名：_____

1. 試求下列 $f(t)$ 的拉普拉斯轉換為何? (20%)

$$(1) f(t) = \sinh at \quad (2) f(t) = \sin bt \cdot \sinh at \quad (3) f(t) = t \sin bt \quad (4) f(t) = \sin^2 bt$$

2. 試求下列 $F(s)$ 的拉普拉斯逆轉換為何? (30%)

$$(1) F(s) = \frac{2s}{s^2 - w^2} \quad (2) F(s) = \frac{6s}{(s^2 + w^2)^2} \quad (3) F(s) = \ln \frac{1}{s-1}$$

$$(4) F(s) = \frac{e^{-2s}}{s^2 + 4s + 8} \quad (5) F(s) = \frac{s^2 + 3s}{s^2 + 9} \quad (6) F(s) = 5e^{-2s}$$

3. 單位步階函數(unit step function)定義為

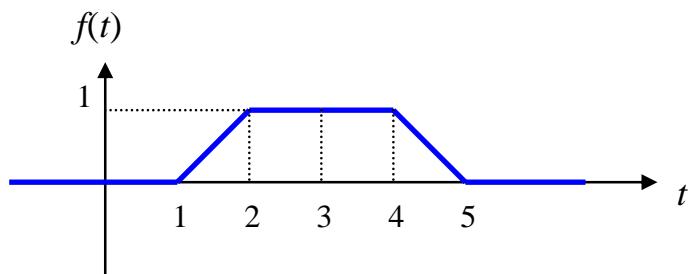
$$u(t-a) = \begin{cases} 1, & t > a \\ 0, & t < a \end{cases}$$

其中 a 為常數

(1) 試畫函數 $f(t) = u(t-1) - u(t-2)$ 之圖形。 (3%)

(2) 並求 $f(t)$ 之拉普拉斯轉換。 (5%)

(3) 試以單位步階函數之組合來表示下圖 $f(t)$ 之函數。 (5%)



(4) 試求 $f(t)$ 之拉普拉斯轉換，即 $F(s) = ?$ (7%)

4. 試以拉普拉斯轉換法求解下述方程式: $y(t) = \cos t + e^{-2t} \int_0^t y(\tau) e^{2\tau} d\tau$ 。 (10%)

5 試以拉普拉斯轉換法求解下述微分方程式: (10%)

$$ty''(t) + 2y'(t) + (2-t)y(t) = 2e^t \quad \text{且} \quad y(0) = 0$$

6. 試求解下述聯立微分方程組 (10%)

$$\begin{cases} \frac{dy_1}{dt} - y_1 - y_2 = 3t \\ \frac{dy_1}{dt} + \frac{dy_2}{dt} - 5y_1 - 2y_2 = 5 \end{cases} \quad \text{且} \quad y_1(0) = 3 \quad \text{與} \quad y_2(0) = 4$$

7. 已知 $F(s) = \frac{1}{(s^2 + 9)^2}$ ，試求 $f(t) = \mathcal{L}^{-1}[F(s)] = ?$ 。 (10%)

8. 上過工數(一)與工數(二)，請問:

(1) 針對工數學習過程或是教學有何心得感想? (5%)

(2) 對於如何幫助學生學好工數這門課，請提供個人看法與建議? (5%)

拉普拉斯轉換： $F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t) e^{-st} dt$

第一平移定理： $\mathcal{L}[e^{at} f(t)] = F(s-a)$

第二平移定理： $\mathcal{L}[f(t-a)u(t-a)] = e^{-as} F(s)$

尺度變換： $\mathcal{L}[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$

微分函數的拉普拉斯轉換： $\mathcal{L}[f'(t)] = sF(s) - f(0)$

$$\mathcal{L}[f''(t)] = s^2 F(s) - sf(0) - f'(0)$$

積分函數的拉普拉斯轉換： $\mathcal{L}\left[\int_0^t f(x) dx\right] = \frac{F(s)}{s}$

$$\mathcal{L}\left[\int_0^t \int_0^\tau f(x) dx d\tau\right] = \frac{F(s)}{s^2}$$

拉普拉斯轉換的微分： $\mathcal{L}[tf(t)] = (-1) \frac{d}{ds} F(s)$

$$\mathcal{L}[t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2} F(s)$$

拉普拉斯轉換的積分： $\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty f(\tau) d\tau$

$$\mathcal{L}\left[\frac{f(t)}{t^2}\right] = \int_s^\infty \int_\gamma^\infty f(\tau) d\tau d\gamma$$

摺積： $f(t) * g(t) = \int_0^t f(\tau) g(t-\tau) d\tau = \int_0^t f(t-\tau) g(\tau) d\tau$

$$\mathcal{L}[f(t) * g(t)] = F(s) \cdot G(s)$$

雙曲函數： $\cosh at = \frac{e^{at} + e^{-at}}{2}$

$$\sinh at = \frac{e^{at} - e^{-at}}{2}$$

初值定理： $\lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} sF(s)$

終值定理： $\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$

一階線性 ODE： $y'(x) + p(x)y(x) = q(x)$ 其積分因子為 $\mu = e^{\int p(x)dx}$

參考解答：

1. 試求下列 $f(t)$ 的拉普拉斯轉換為何? (20%)

(1) $f(t) = \sinh at$ (2) $f(t) = \sin bt \cdot \sinh at$ (3) $f(t) = t \sin bt$ (4) $f(t) = \sin^2 bt$

(1) $f(t) = \sinh at = \frac{e^{at} - e^{-at}}{2}$

$$\therefore \mathcal{L}[\sinh at] = \frac{1}{2} \mathcal{L}[e^{at} - e^{-at}] = \frac{1}{2} \left(\frac{1}{s-a} - \frac{1}{s+a} \right) = \frac{a}{s^2 - a^2}$$

(2) $f(t) = \sin bt \cdot \sinh at = \sin bt \cdot \frac{e^{at} - e^{-at}}{2} = \frac{1}{2} (e^{at} \sin bt - e^{-at} \sin bt)$

$$\text{又 } \mathcal{L}[\sin bt] = \frac{b}{s^2 + b^2}$$

$$\Rightarrow \mathcal{L}[e^{at} \sin bt] = \frac{b}{(s-a)^2 + b^2} \text{ 與 } \mathcal{L}[e^{-at} \sin bt] = \frac{b}{(s+a)^2 + b^2}$$

$$\therefore \mathcal{L}[\sin bt \cdot \sinh at] = \frac{1}{2} \left[\frac{b}{(s-a)^2 + b^2} - \frac{b}{(s+a)^2 + b^2} \right]$$

(3) $\mathcal{L}[\sin at] = \frac{a}{s^2 + a^2}$

$$\therefore \mathcal{L}[t \sin at] = -\frac{d}{ds} \left(\frac{a}{s^2 + a^2} \right) = \frac{2as}{(s^2 + a^2)^2}$$

(4) $f(t) = \sin^2 bt = \frac{1 - \cos 2bt}{2}$

$$\therefore \mathcal{L}[\sin^2 bt] = \frac{1}{2} \left(\frac{1}{s} - \frac{s}{s^2 + 4b^2} \right) = \frac{2b^2}{s(s^2 + 4b^2)}$$

2. 試求下列 $F(s)$ 的拉普拉斯逆轉換為何? (30%)

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(5) $F(s) = \frac{s^2 + 3s}{s^2 + 9}$

(6) $F(s) = 5e^{-2s}$

(1) $F(s) = \frac{2s}{s^2 - w^2}$

$$\therefore f(t) = \mathcal{L}^{-1} \left[\frac{2s}{s^2 - w^2} \right] = 2 \cosh wt$$

(2) $F(s) = \frac{6s}{(s^2 + w^2)^2}$

$$\therefore f(t) = \mathcal{L}^{-1}\left[\frac{6s}{(s^2 + w^2)^2}\right] = -\frac{3}{w} \frac{d}{ds} \left(\frac{w}{s^2 + w^2}\right) = \frac{3t}{w} \sin wt$$

$$(3) F(s) = \ln \frac{1}{s-1} = \ln 1 - \ln(s-1) = -\ln(s-1)$$

$$\mathcal{L}[f(t)] = F(s) = -\ln(s-1)$$

$$\mathcal{L}[t \cdot f(t)] = -\frac{d}{ds} F(s) = \frac{1}{s-1}$$

$$\therefore t \cdot f(t) = \mathcal{L}^{-1}\left[\frac{1}{s-1}\right] = e^t \Rightarrow f(t) = \frac{e^t}{t}$$

$$(4) \mathcal{L}^{-1}\left[\frac{1}{s^2 + 4s + 8}\right] = \mathcal{L}^{-1}\left[\frac{1}{(s+2)^2 + 2^2}\right] = \frac{1}{2} e^{-2t} \mathcal{L}^{-1}\left[\frac{2}{s^2 + 2^2}\right] = \frac{1}{2} e^{-2t} \sin 2t$$

$$\therefore f(t) = \mathcal{L}^{-1}\left[\frac{e^{-2s}}{(s+2)^2 + 2^2}\right] = \frac{1}{2} e^{-2(t-2)} \cdot \sin 2(t-2) \cdot u(t-2)$$

$$(5) F(s) = \frac{s^2 + 3s}{s^2 + 9} = 1 + \frac{3s}{s^2 + 9} - \frac{9}{s^2 + 9}$$

$$\therefore \mathcal{L}^{-1}[F(s)] = \delta(t) + 3 \cos 3t - 3 \sin 3t$$

$$(6) F(s) = 5e^{-2s}$$

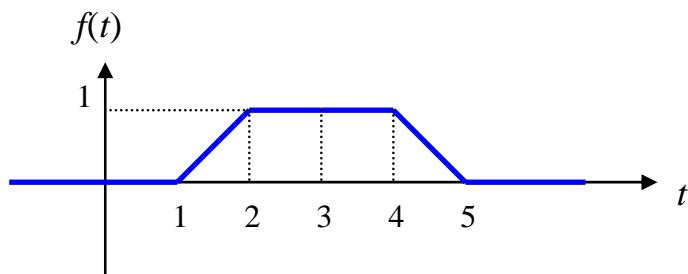
$$\therefore \mathcal{L}^{-1}[F(s)] = 5\delta(t-2)$$

3. 單位步階函數(unit step function)定義為

$$u(t-a) = \begin{cases} 1, & t > a \\ 0, & t < a \end{cases}$$

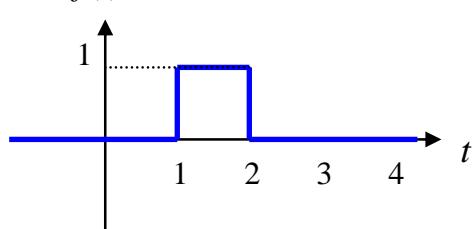
其中 a 為常數

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- (4) 試求 $f(t)$ 之拉普拉斯轉換，即 $F(s) = ?$ (7%)

$$(1)$$



$$(2) \quad F(s) = \mathcal{L}[u(t-a)] = \int_0^\infty u(t-a) e^{-st} dt = \int_a^\infty e^{-st} dt = \frac{1}{s} e^{-as}$$

$$\mathcal{L}[f(t)] = \mathcal{L}[u(t-1) - u(t-2)] = \frac{1}{s}(e^{-s} - e^{-2s})$$

(3)

$$\begin{aligned} f(t) &= (t-1) \cdot [u(t-1) - u(t-2)] + [u(t-2) - u(t-4)] + (5-t)[u(t-4) - u(t-5)] \\ &= (t-1) \cdot u(t-1) - (t-2) \cdot u(t-2) - (t-4) \cdot u(t-4) + (t-5) \cdot u(t-5) \end{aligned}$$

$$(4) \quad \mathcal{L}[f(t)] = \frac{1}{s^2} e^{-s} - \frac{1}{s^2} e^{-2s} - \frac{1}{s^2} e^{-4s} + \frac{1}{s^2} e^{-5s} = \frac{1}{s^2}(e^{-s} - e^{-2s} - e^{-4s} + e^{-5s})$$

4. 試以拉普拉斯轉換法求解下述方程式: $y(t) = \cos t + e^{-2t} \int_0^t y(\tau) e^{2\tau} d\tau$ (10%)

$$\mathcal{L}[y(t)] = Y(s)$$

$$\text{又 } \mathcal{L}[e^{-2t} \int_0^t y(\tau) e^{2\tau} d\tau] = \mathcal{L}\left[\int_0^t y(\tau) e^{-2(t-\tau)} d\tau\right] = \mathcal{L}[y(t) * e^{-2t}] = Y(s) \cdot \frac{1}{s+2}$$

$$\therefore \mathcal{L}[y(t)] = \mathcal{L}[\cos t + e^{-2t} \int_0^t y(\tau) e^{2\tau} d\tau]$$

$$\Rightarrow Y(s) = \frac{s}{s^2 + 1} + Y(s) \cdot \frac{1}{s+2}$$

$$\Rightarrow \left(\frac{s+1}{s+2}\right)Y(s) = \frac{s}{s^2 + 1}$$

$$\Rightarrow Y(s) = \frac{s}{s^2 + 1} \cdot \frac{s+2}{s+1} = \frac{1}{2} \left(\frac{3s+1}{s^2+1} - \frac{1}{s+1} \right) = \frac{1}{2} \left(\frac{3s}{s^2+1} + \frac{1}{s^2+1} - \frac{1}{s+1} \right)$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}[Y(s)] = \frac{1}{2}(3\cos t + \sin t - e^{-t})$$

5 試以拉普拉斯轉換法求解下述微分方程式: (10%)

$$ty''(t) + 2y'(t) + (2-t)y(t) = 2e^t \quad \text{且} \quad y(0) = 0$$

$$\mathcal{L}[ty''(t) + 2y'(t) + (2-t)y(t)] = \mathcal{L}[2e^t]$$

$$\Rightarrow -\frac{d}{ds}[s^2 Y(s) - sy(0) - y'(0)] + 2[sY(s) - y(0)] + 2Y(s) + \frac{d}{ds}Y(s) = \frac{2}{s-1}$$

$$\Rightarrow (1-s^2)Y'(s) + 2Y(s) = \frac{2}{s-1}$$

$$\Rightarrow Y'(s) + \frac{2}{1-s^2}Y(s) = \frac{2}{(s-1)(1-s^2)} \quad \text{此為一階線性 ODE}$$

$$\text{積分因子為 } \mu = e^{\int \frac{2}{1-s^2} dx} = e^{-\int (\frac{1}{s-1} - \frac{1}{s+1}) dx} = e^{\ln(s+1) - \ln(s-1)} = \frac{s+1}{s-1}$$

$$\text{同乘積分因子後可得 } \frac{s+1}{s-1}Y'(s) + \frac{s+1}{s-1} \cdot \frac{2}{1-s^2}Y(s) = \frac{s+1}{s-1} \cdot \frac{2}{(s-1)(1-s^2)}$$

$$\begin{aligned} &\Rightarrow \frac{d}{ds} \left[\frac{s+1}{s-1} Y(s) \right] = -\frac{2}{(s-1)^3} \\ &\Rightarrow \frac{s+1}{s-1} Y(s) = -\int \frac{2}{(s-1)^3} ds = \frac{1}{(s-1)^2} + c \\ &\Rightarrow Y(s) = \frac{1}{s^2-1} + c \frac{s-1}{s+1} \end{aligned}$$

由初值定理 $\lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} sF(s)$ 可知 $c = 0$

$$\therefore Y(s) = \frac{1}{s^2-1} \Rightarrow y(t) = \mathcal{L}^{-1}[Y(s)] = \sinh t$$

6. 試求解下述聯立微分方程組 (10%)

$$\begin{cases} \frac{dy_1}{dt} - y_1 - y_2 = 3t \\ \frac{dy_1}{dt} + \frac{dy_2}{dt} - 5y_1 - 2y_2 = 5 \end{cases} \quad \text{且 } y_1(0) = 3 \text{ 與 } y_2(0) = 4$$

$$\mathcal{L}[y_1(t)] = Y_1(s) \text{ 與 } \mathcal{L}[y_2(t)] = Y_2(s)$$

將聯立微分方程組 $\begin{cases} \frac{dy_1}{dt} - y_1 - y_2 = 3t \\ \frac{dy_1}{dt} + \frac{dy_2}{dt} - 5y_1 - 2y_2 = 5 \end{cases}$ 做拉普拉斯轉換後可得

$$\begin{cases} sY_1(s) - y_1(0) - Y_1(s) - Y_2(s) = \frac{3}{s^2} \\ sY_1(s) - y_1(0) + sY_2(s) - y_2(0) - 5Y_1(s) - 2Y_2(s) = \frac{5}{s} \end{cases}$$

$$\Rightarrow \begin{cases} (s-1)Y_1(s) - Y_2(s) = \frac{3}{s^2} + 3 \\ (s-5)Y_1(s) + (s-2)Y_2(s) = \frac{5}{s} + 7 \end{cases}$$

$$\Rightarrow Y_1(s) = \frac{3s^3 + s^2 + 8s - 6}{s^2(s+1)(s-3)}, \quad Y_2(s) = \frac{4s^3 + 13s^2 - 8s + 15}{s^2(s+1)(s-3)}$$

$$\text{由 } Y_1(s) = \frac{3s^3 + s^2 + 8s - 6}{s^2(s+1)(s-3)} = \frac{As+B}{s^2} + \frac{C}{s-3} + \frac{D}{s+1}$$

$$\text{通分後可得 } (As+B)(s-3)(s+1) + Cs^2(s+1) + Ds^2(s-3) = 3s^3 + s^2 + 8s - 6$$

$$\text{當 } s=0 \Rightarrow -3B=-6 \Rightarrow B=2$$

$$\text{當 } s=3 \Rightarrow 36C=108 \Rightarrow C=3$$

$$\text{當 } s=-1 \Rightarrow -4D=-16 \Rightarrow D=4$$

$$\text{比較 } s^3 \text{ 項可知: } A+C+D=3 \Rightarrow A=-4$$

$$y_1(t) = \mathcal{L}^{-1}[Y_1(s)] = \mathcal{L}^{-1}\left[\frac{-4s+2}{s^2} + \frac{3}{s-3} + \frac{4}{s+1}\right] = -4 + 2t + 3e^{3t} + 4e^{-t}$$

同理可得：

$$y_2(t) = \mathcal{L}^{-1}[Y_2(s)] = \mathcal{L}^{-1}\left[\frac{6s-5}{s^2} + \frac{6}{s-3} + \frac{-8}{s+1}\right] = 6 - 5t + 6e^{3t} - 8e^{-t}$$

7. 已知 $F(s) = \frac{1}{(s^2 + 9)^2}$ ，試求 $f(t) = \mathcal{L}^{-1}[F(s)] = ?$ 。(10%)

$$F(s) = \frac{1}{s^2 + 9} \cdot \frac{1}{s^2 + 9}$$

$$\therefore \text{令 } H(s) = G(s) = \frac{1}{s^2 + a^2}$$

$$\Rightarrow h(t) = g(t) = \mathcal{L}^{-1}[H(s)] = \mathcal{L}^{-1}\left[\frac{1}{s^2 + 3^2}\right] = \frac{1}{3} \mathcal{L}^{-1}\left[\frac{3}{s^2 + 3^2}\right] = \frac{1}{3} \sin 3t$$

$$f(t) = h(t) * g(t)$$

$$= \frac{1}{9} \int_0^t \sin 3\tau \cdot \sin 3(t-\tau) d\tau$$

$$= \frac{1}{9} \int_0^t \sin 3\tau \cdot (\sin 3t \cos 3\tau - \cos 3t \sin 3\tau) d\tau$$

$$= \frac{1}{9} \int_0^t \left(\frac{1}{2} \sin 3t \sin 6\tau - \cos 3t \sin^2 3\tau \right) d\tau$$

$$= \frac{1}{18} \int_0^t [\sin 3t \sin 6\tau - \cos 3t(1 - \cos 6\tau)] d\tau$$

$$= \frac{1}{18} \left[-\frac{1}{6} \sin 3t \cos 6\tau - \cos 3t \left(\tau - \frac{1}{6} \sin 6\tau \right) \right]_0^t$$

$$= \frac{1}{18} \left(-\frac{1}{6} \sin 3t \cos 6t + \frac{1}{6} \sin 3t - t \cos 3t + \frac{1}{6} \sin 6t \cos 3t \right)$$

$$= \frac{1}{18} \left(\frac{1}{3} \sin 3t - t \cos 3t \right)$$

$$= \frac{\sin 3t - 3t \cos 3t}{54}$$