國立臺灣海洋大學河海工程學系 2018 工程數學(二) 第一次大考參考解答

系級：__________　　學號：__________　　姓名：__________

1. 已知 $f = xz - yz$，$\vec{A} = y^2\vec{i} + (y^2 - x^2)\vec{j} + 2z^2\vec{k}$，試求:

   (1) $\nabla^2(xzf)$ (2) $\nabla\cdot(\nabla f)$ (3) $\nabla\times\vec{A}$ (4) $\nabla(\nabla\cdot\vec{A})$ (12%)

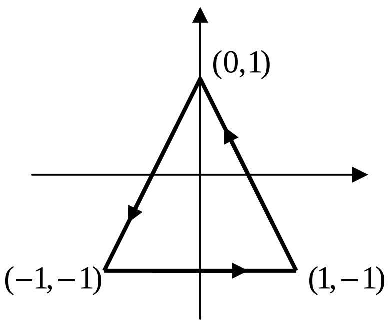
2. 已知一質點的運動軌跡為 $\vec{r}(t) = t^2\,\hat{i} + \sin t\,\hat{j} + \cos t\,\hat{k}$，試求其速度、速率、加速度、切線加速度與法線加速度。 (10%)

3. 某應力場 $S(x,y,z)$ 之大小與 (x,y,z) 距座標原點之距離成反比。已知 $S(1,1,1) = 10\sqrt{3}$，B 點座標為 $B(-1,3,4)$，A 點座標為 $A(1,2,2)$，求 $S(x,y,z)$ 在 A 點沿著 $\overrightarrow{AB}$ 之應力變化率。 (10%)

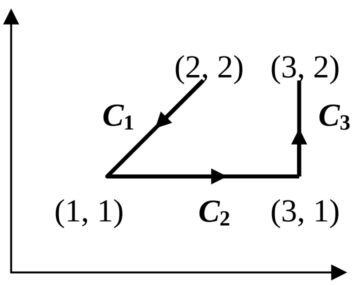
4. 已知一 2 維空間向量場

$$\vec{F}(x,y) = \left(\frac{-y}{x^2+y^2} + x^2\right)\vec{i} + \left(\frac{x}{x^2+y^2} - 2y\right)\vec{j}$$

   求閉合曲線線積分 $I = \oint_C \vec{F}\cdot d\vec{r}$ 之值，其中 C 為三角形逆時鐘閉合曲線如圖一所示，$\vec{r}$ 表示 C 之位置向量。 (10%)



圖一



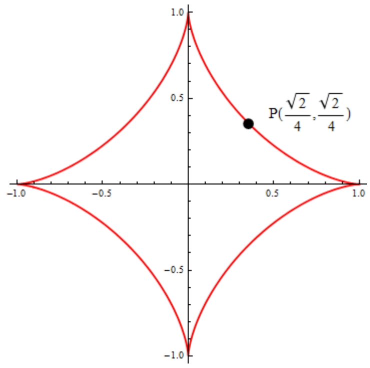
圖二

5. 試求線積分 $I = \int_C 2xydx + (x^2 - 3y^2)dy$，其中曲線 C 如圖二所示。 (10%)

6. 平面曲線方程式 $\Gamma: x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$ 與曲線上一點 $P(\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4})$ 如圖三所示。

   (1) 請計算曲線繞一圈之總弧長。 (5%)

   (2) 求曲線上 P 點之曲率。 (5%)



圖三

7. 請參考下圖，並回答下列各題：

其中，$\vec{F}=x\vec{i}+(2z-x)\vec{j}-y^2\vec{k}$，$S=S_1+S_2+S_3+S_4$，

$\Gamma=\Gamma_1+\Gamma_2+\Gamma_3$

(1) $\vec{F}$ **是否為保守場？請說明之。(5%)**

(2) $\vec{n}_4$ **為斜面 S_4 上的單位法向量，試問：** $\vec{n}_4=?$ **並求斜面 S_4 的方程式。(8%)**

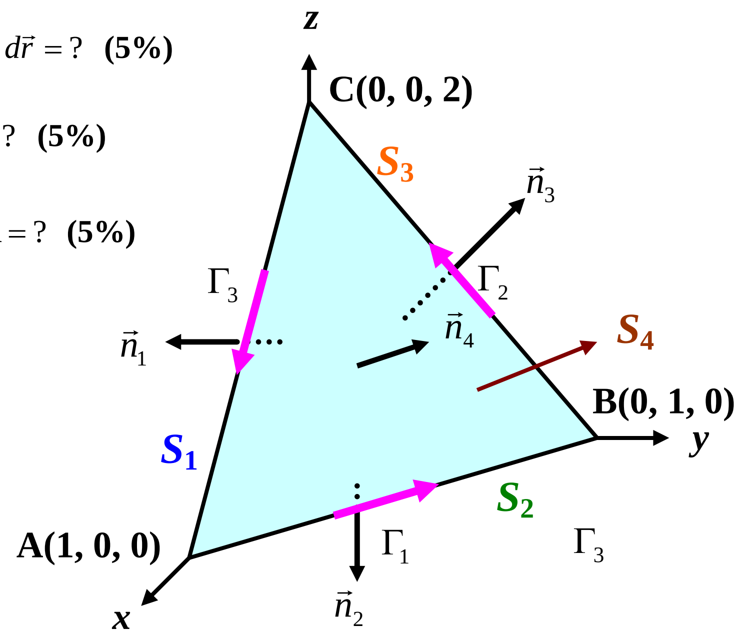
(3) 斜面 S_4 的面積為何？(5%)

(4) $\oiint \vec{F}\cdot\vec{n}\,dS=?$ **(5%)**

(5) 請使用線積分計算 $\int_{\Gamma}\vec{F}\cdot d\vec{r}=?$ **(5%)**

(6) 請計算 $\iint_{S_4}(\nabla\times\vec{F})\cdot\vec{n}\,dA=?$ **(5%)**

(7) 請計算 $\iint_{S_1+S_2+S_3}(\nabla\times\vec{F})\cdot\vec{n}\,dA=?$ **(5%)**



Hint:

Gauss 散度定理： $\iiint \nabla\cdot\vec{F}\,dV=\oiint \vec{F}\cdot\vec{n}\,dA$ **(3D)**

$\iint \nabla\cdot\vec{F}\,dA=\oint \vec{F}\cdot\vec{n}\,ds$ **(2D)**

格林定理： $\int P\,dx+Q\,dy=\iint(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})\,dxdy$

Stokes 旋度定理： $\iint(\nabla\times\vec{F})\cdot\vec{n}\,dA=\oint\vec{F}\cdot d\vec{r}$

曲率： $\kappa=\frac{|y''|}{[1+(y')^2]^{\frac{3}{2}}}$ **扭率：** $\tau=\left|\frac{d(\vec{T}(s)\times\vec{N}(s))}{ds}\right|$

二倍角公式： $\cos^2\theta=\frac{1+\cos 2\theta}{2}$，$\sin^2\theta=\frac{1-\cos 2\theta}{2}$

四面體體積： $V=\frac{1}{3}A_0h$，**(**A_0**：底面積；**h**：高)**

參考解答:

1. (1) $\nabla^2(xz\,f)=\dfrac{\partial^2(x^2z^2-xyz^2)}{\partial x^2}+\dfrac{\partial^2(x^2z^2-xyz^2)}{\partial y^2}+\dfrac{\partial^2(x^2z^2-xyz^2)}{\partial z^2}$

$=2x^2+2z^2-2xy$

(2) $\nabla\cdot(\nabla f)=\nabla^2 f=\dfrac{\partial^2(xz-yz)}{\partial x^2}+\dfrac{\partial^2(xz-yz)}{\partial y^2}+\dfrac{\partial^2(xz-yz)}{\partial z^2}=0$

(3) $\nabla\times\vec{A}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ y^2 & y^2-x^2 & 2z^2\end{vmatrix}=-2(x+y)\vec{k}$

(4) $\nabla(\nabla\cdot\vec{A})=\nabla(\dfrac{\partial(y^2)}{\partial x}+\dfrac{\partial(y^2-x^2)}{\partial y}+\dfrac{\partial(2z^2)}{\partial z})$

$=\nabla(2y+4z)$

$=2\vec{j}+4\vec{k}$

2. 速度: $\vec{v}(t)=\vec{r}\,'(t)=2t\,\hat{i}+\cos t\,\hat{j}-\sin t\,\hat{k}$

速率: $v(t)=\left|\vec{v}(t)\right|=\sqrt{(2t)^2+(\cos t)^2+(\sin t)^2}=\sqrt{4t^2+1}$

加速度: $\vec{a}(t)=\vec{v}\,'(t)=\vec{r}\,''(t)=2\hat{i}-\sin t\,\hat{j}-\cos t\,\hat{k}$

切線加速度: $\vec{a}_t(t)=(\vec{a}\cdot\dfrac{\vec{v}}{|\vec{v}|})\dfrac{\vec{v}}{|\vec{v}|}=\dfrac{8t^2}{4t^2+1}\hat{i}+\dfrac{4t\cos t}{4t^2+1}\hat{j}-\dfrac{4t\sin t}{4t^2+1}\hat{k}$

法線加速度: $\vec{a}_n(t)=\vec{a}-\vec{a}_t\hat{k}$

$=\dfrac{2}{4t^2+1}\hat{i}+(-\sin t-\dfrac{4t\cos t}{4t^2+1})\hat{j}+(-\cos t+\dfrac{4t\sin t}{4t^2+1})$

3. $S(x,y,z)=\dfrac{k}{\sqrt{x^2+y^2+z^2}}$ 又 $S(1,1,1)=10\sqrt{3}$

$\therefore$ 可知 $k=30$,即 $S(x,y,z)=\dfrac{30}{\sqrt{x^2+y^2+z^2}}$

$\overrightarrow{AB}=(-1,3,4)-(1,2,2)=(-2,1,2)=-2\vec{i}+\vec{j}+2\vec{k}$

$\vec{u}_{AB}=\dfrac{-2\vec{i}+\vec{j}+2\vec{k}}{\sqrt{(-2)^2+1^2+2^2}}=-\dfrac{2}{3}\vec{i}+\dfrac{1}{3}\vec{j}+\dfrac{2}{3}\vec{k}$

又 $\nabla S = \dfrac{-30(x\hat{i} + y\hat{j} + z\hat{k})}{(x^2+y^2+z^2)^{\frac{3}{2}}}$ $\Rightarrow$ 在 A 點之 $\nabla S = \dfrac{-10(\hat{i} + 2\hat{j} + 2\hat{k})}{9}$

$S(x,y,z)$ 在 A 點沿著 $\overrightarrow{AB}$ 之應力變化率為 $\nabla S \cdot \vec{u}_{AB} = -\dfrac{40}{27}$

4. $I = \oint_C \vec{F} \cdot d\vec{r} = \oint_C (\dfrac{-y}{x^2+y^2} + x^2)\,dx + (\dfrac{x}{x^2+y^2} - 2y)\,dy$

$$= \iint_A [\frac{\partial}{\partial x}(\frac{x}{x^2+y^2}) - \frac{\partial}{\partial y}(\frac{-y}{x^2+y^2} + x^2)]dxdy$$

$$= 0$$

5. C_1: $(2, 2) \to (1, 1)$ $\Rightarrow$ 令 $x = t$, $y = t$ $(t: 2 \to 1)$ $\Rightarrow dx = dt$, $dy = dt$

$$\Rightarrow \int_{C_1} 2xydx + (x^2 - 3y^2)dy = \int_2^1 (2t^2 + t^2 - 3t^2)dt = 0$$

C_2: $(1, 1) \to (3, 1)$ $\Rightarrow$ $x: 1 \to 3$, $y = 1$ $\Rightarrow$ $dy = 0$

$$\Rightarrow \int_{C_2} 2xydx + (x^2 - 3y^2)dy = \int_1^3 2x\,dx = 8$$

C_3: $(3, 1) \to (3, 2)$ $\Rightarrow$ $x = 3$, $y: 1 \to 2$ $\Rightarrow$ $dx = 0$

$$\Rightarrow \int_{C_3} 2xydx + (x^2 - 3y^2)dy = \int_1^2 (9 - 3y^2)\,dy = 2$$

$$I = \int_C 2xydx + (x^2 - 3y^2)dy = \int_{C_1} + \int_{C_2} + \int_{C_3} = 10$$

6. (1) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$ $\Rightarrow y = (1 - x^{\frac{2}{3}})^{\frac{3}{2}}$

$$y' = \frac{dy}{dx} = \frac{3}{2}(1 - x^{\frac{2}{3}})^{\frac{1}{2}} \cdot (-\frac{2}{3}x^{-\frac{1}{3}}) = -(1 - x^{\frac{2}{3}})^{\frac{1}{2}} \cdot x^{-\frac{1}{3}}$$

$$y'' = \frac{d^2y}{dx^2} = -\left[\frac{1}{2}(1 - x^{\frac{2}{3}})^{-\frac{1}{2}} \cdot (-\frac{2}{3}x^{-\frac{1}{3}}) \cdot x^{-\frac{1}{3}} + (1 - x^{\frac{2}{3}})^{\frac{1}{2}} \cdot (-\frac{1}{3}x^{-\frac{4}{3}})\right]$$

$$= \frac{1}{3}\left[(1 - x^{\frac{2}{3}})^{-\frac{1}{2}} \cdot (x^{-\frac{2}{3}}) + (1 - x^{\frac{2}{3}})^{\frac{1}{2}} \cdot (x^{-\frac{4}{3}})\right] = \frac{1}{3(1 - x^{\frac{2}{3}})^{\frac{1}{2}} x^{\frac{4}{3}}}$$

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + (\frac{dy}{dx})^2}\, dx = \sqrt{x^{-\frac{2}{3}}}\, dx = x^{-\frac{1}{3}}dx$$

$$s = 4\int_0^1 x^{-\frac{1}{3}}dx = 6x^{\frac{2}{3}}\,|_0^1 = 6$$

(2) $\kappa = \dfrac{|y''|}{[1+(y')^2]^{\frac{3}{2}}} = \dfrac{1}{3(1-x^{\frac{2}{3}})^{\frac{1}{2}}x^{\frac{1}{3}}}$

又 $P(\dfrac{\sqrt{2}}{4}, \dfrac{\sqrt{2}}{4}) \Rightarrow \kappa = \dfrac{2}{3}$

7. (1) $\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & 2z-x & -y^2 \end{vmatrix} = (-2y-2)\vec{i} - \vec{k}$ $\Rightarrow$ 此為非保守場

(2) $\vec{a} = (0, 1, 0) - (1, 0, 0) = (-1, 1, 0)$

$\vec{b} = (0, 0, 2) - (1, 0, 0) = (-1, 0, 2)$

$\vec{a} \times \vec{b} = (2, 2, 1)$

$\vec{n}_4 = \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} = \dfrac{(2, 2, 1)}{\sqrt{2^2+2^2+1^2}} = (\dfrac{2}{3}, \dfrac{2}{3}, \dfrac{1}{3})$

平面方程式為 $(x-1, y, z) \cdot (\dfrac{2}{3}, \dfrac{2}{3}, \dfrac{1}{3}) = 0 \Rightarrow 2x + 2y + z = 2$

(3) 斜面 S_4 的面積=$\dfrac{1}{2}|\vec{a} \times \vec{b}| = \dfrac{3}{2}$

(4) 由 Gauss 散度定理: $\oiint \vec{F} \cdot \vec{n}\, dS = \iiint \nabla \cdot \vec{F}\, dV = \iiint dV = \dfrac{1}{3} \cdot \dfrac{1}{2} \cdot 2 = \dfrac{1}{3}$

(5) Γ_1: $x + y = 1 \Rightarrow y = 1 - x \Rightarrow dy = -dx$ $(z = 0)$

Γ_2: $2y + z = 2 \Rightarrow z = 2 - 2y \Rightarrow dz = -2dy$ $(x = 0)$

Γ_3: $2x + z = 2 \Rightarrow z = 2 - 2x \Rightarrow dz = -2dx$ $(y = 0)$

$$\begin{aligned}
\int_{\Gamma} \vec{F} \cdot d\vec{r} &= \int_{\Gamma_1} \vec{F} \cdot d\vec{r} + \int_{\Gamma_2} \vec{F} \cdot d\vec{r} + \int_{\Gamma_3} \vec{F} \cdot d\vec{r} \\
&= \int_{\Gamma_1} xdx + (2z-x)dy + \int_{\Gamma_2} (2z-x)dy - y^2 dz \\
&\quad + \int_{\Gamma_3} xdx - y^2 dz \\
&= \int_1^0 2xdx + \int_1^0 (4 - 4y + 2y^2)dy + \int_0^1 xdx \\
&= -\frac{19}{6}
\end{aligned}$$

(6) 由 Stokes 旋度定理: $\iint_{S_4} (\nabla \times \vec{F}) \cdot \vec{n}\, dA = \oint_{\Gamma} \vec{F} \cdot d\vec{r} = -\dfrac{19}{6}$

(7) 由 Stokes 旋度定理: $\iint_{S_1+S_2+S_3} (\nabla \times \vec{F}) \cdot \vec{n}\, dA = -\oint_{\Gamma} \vec{F} \cdot d\vec{r} = \dfrac{19}{6}$