# 國立臺灣海洋大學河海工程學系 2018 工程數學(二) 期末考參考解答

系級：______________ 學號：____________ 姓名：___________

1. 試求下列 $f(t)$ 的拉普拉斯轉換為何? (20%)

   (1) $f(t)=\sin t\cosh 2t$ (2) $f(t)=t\sinh 2t$ (3) $f(t)=\cos^2 t$ (4) $f(t)=\cos^3 t$

2. 試求下列 $F(s)$ 的拉普拉斯逆轉換為何? (25%)

   (1) $F(s)=\dfrac{1}{s(s^2+w^2)}$ (2) $F(s)=\dfrac{1}{s^2(s^2+w^2)}$ (3) $F(s)=\dfrac{s}{(s^2+w^2)^2}$

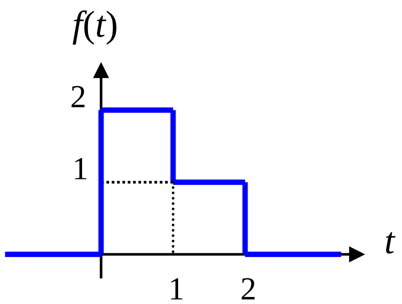
   (4) $F(s)=\dfrac{1}{s^2+w^2+2ws}$ (5) $F(s)=\dfrac{1}{s^2+w^2+ws}$

3. (1) 單位步階函數(unit step function)定義為

$$u(t-a)=\begin{cases}1, & t>a\\ 0, & t<a\end{cases}$$

   其中 $a$ 為常數，試求 $u(t-5)$ 之拉普拉斯轉換。 (5%)

   (2) 試以單位步階函數之組合來表示下圖 $f(t)$ 之函數。 (5%)




   (3) 試求 $f(t)$ 之拉普拉斯轉換，即 $F(s)=?$ (5%)

4. 試以拉普拉斯轉換法求解下述方程式。 (20%)

   (1) $y'(t)=\cos t+\int_0^t y(\tau)\cos(t-\tau)\,d\tau$ 且 $y(0)=1$

   (2) $y''(t)+3y'(t)+2y(t)=3\delta(x-2)$ 且 $y'(0)=y(0)=0$

5. 試求解下述聯立微分方程組 (10%)

$$\begin{cases}\dfrac{dy_1}{dt}=-y_1-y_2\\ \dfrac{dy_2}{dt}=y_1-y_2\end{cases}$$ 且 $y_1(0)=0$ 與 $y_2(0)=1$

6. 若 $f(t)=2\sin 2t$ ， $g(t)=\cos 2t$ ，試求 $f(t)*g(t)$ 。(10%)

拉普拉斯轉換： $F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)\, e^{-st}\, dt$

第一平移定理： $\mathcal{L}[e^{at} f(t)] = F(s-a)$

第二平移定理： $\mathcal{L}[f(t-a)u(t-a)] = e^{-as} F(s)$

尺度變換： $\mathcal{L}[f(at)] = \frac{1}{a} F(\frac{s}{a})$

微分函數的拉普拉斯轉換： $\mathcal{L}[f'(t)] = sF(s) - f(0)$

$$\mathcal{L}[f''(t)] = s^2 F(s) - sf(0) - f'(0)$$

積分函數的拉普拉斯轉換： $\mathcal{L}[\int_0^t f(x)\, dx] = \frac{F(s)}{s}$

$$\mathcal{L}[\int_0^t \int_0^t f(x)\, dx\, dx\,] = \frac{F(s)}{s^2}$$

拉普拉斯轉換的微分： $\mathcal{L}[tf(t)] = (-1)\frac{d}{ds} F(s)$

$$\mathcal{L}[t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2} F(s)$$

拉普拉斯轉換的積分： $\mathcal{L}[\frac{f(t)}{t}] = \int_s^\infty f(\tau)\, d\tau$

$$\mathcal{L}[\frac{f(t)}{t^2}] = \int_s^\infty \int_\gamma^\infty f(\tau)\, d\tau\, d\gamma$$

摺積： $f(t) * g(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$

雙曲函數： $\cosh at = \frac{e^{at} + e^{-at}}{2}$

$$\sinh at = \frac{e^{at} - e^{-at}}{2}$$

參考解答：

**1. 試求下列 $f(t)$ 的拉普拉斯轉換為何? (20%)**

**(1)** $f(t)=\sin t\cosh 2t$ **(2)** $f(t)=t\sinh 2t$ **(3)** $f(t)=\cos^2 t$ **(4)** $f(t)=\cos^3 t$

(1) $f(t)=\sin t\cosh 2t=\sin t\cdot\dfrac{e^{2t}+e^{-2t}}{2}=\dfrac{1}{2}(e^{2t}\sin t+e^{-2t}\sin t)$

又 $\mathcal{L}[\sin t]=\dfrac{1}{s^2+1}$

$\therefore \mathcal{L}[e^{2t}\sin t]=\dfrac{1}{(s-2)^2+1}$ 與 $\mathcal{L}[e^{-2t}\sin t]=\dfrac{1}{(s+2)^2+1}$

$\mathcal{L}[\sin t\cosh 2t]=\dfrac{1}{2}[\dfrac{1}{(s-2)^2+1}+\dfrac{1}{(s+2)^2+1}]$

(2) $\mathcal{L}[t]=\dfrac{1}{s^2}$

$\therefore \mathcal{L}[t\sinh 2t]=\dfrac{1}{2}[\dfrac{1}{(s-2)^2}-\dfrac{1}{(s+2)^2}]=\dfrac{4s}{(s^2-4)^2}$

(3) $f(t)=\cos^2 t=\dfrac{1+\cos 2t}{2}$

$\therefore \mathcal{L}[\cos^2 t]=\dfrac{1}{2}(\dfrac{1}{s}+\dfrac{s}{s^2+4})=\dfrac{s^2+2}{s(s^2+4)}$

(4) $\cos 3t=4\cos^3 t-3\cos t \Rightarrow \cos^3 t=\dfrac{\cos 3t+3\cos t}{4}$

$\therefore \mathcal{L}[\cos^3 t]=\dfrac{1}{4}(\dfrac{s}{s^2+9}+\dfrac{3s}{s^2+1})=\dfrac{s^3+7s}{(s^2+1)(s^2+9)}$

**2. 試求下列 $F(s)$ 的拉普拉斯逆轉換為何? (25%)**

**(1)** $F(s)=\dfrac{1}{s(s^2+w^2)}$ **(2)** $F(s)=\dfrac{1}{s^2(s^2+w^2)}$ **(3)** $F(s)=\dfrac{1}{(s^2+w^2)^2}$

**(4)** $F(s)=\dfrac{1}{s^2+w^2+2ws}$ **(5)** $F(s)=\dfrac{1}{s^2+w^2+ws}$

(1) $F(s)=\dfrac{1}{s(s^2+w^2)}=\dfrac{1}{w^2}(\dfrac{1}{s}-\dfrac{s}{s^2+w^2})$

$\therefore \mathcal{L}^{-1}[F(s)]=\dfrac{1}{w^2}(1-\cos wt)$

(2) $F(s)=\dfrac{1}{s^2(s^2+w^2)}=\dfrac{1}{w^2}(\dfrac{1}{s^2}-\dfrac{1}{s^2+w^2})$

$\therefore \mathcal{L}^{-1}[F(s)]=\dfrac{1}{w^2}(t-w\sin wt)$

(3) $F(s)=\frac{s}{(s^2+w^2)^2}=-\frac{1}{2}\frac{d}{ds}(\frac{1}{s^2+w^2})$

$\therefore \mathcal{L}^{-1}[F(s)]=\frac{1}{2w}t\sin wt$

(4) $F(s)=\frac{1}{s^2+w^2+2ws}=\frac{1}{(s+w)^2}$

$\therefore \mathcal{L}^{-1}[F(s)]=\mathcal{L}^{-1}[\frac{1}{(s+w)^2}]=e^{-wt}\cdot\mathcal{L}^{-1}[\frac{1}{s^2}]=te^{-wt}$

(5) $F(s)=\frac{1}{s^2+w^2+ws}=\frac{1}{(s+\frac{w}{2})^2+(\frac{\sqrt{3}}{2}w)^2}$

$$\therefore \mathcal{L}^{-1}[F(s)]=\mathcal{L}^{-1}[\frac{1}{(s+\frac{w}{2})^2+(\frac{\sqrt{3}}{2}w)^2}]$$

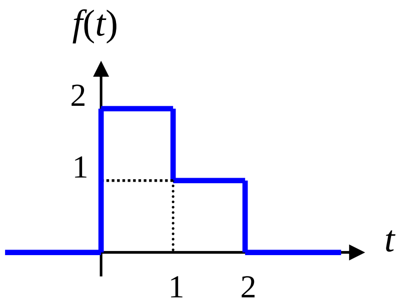
$$=e^{-\frac{1}{2}wt}\cdot\mathcal{L}^{-1}[\frac{1}{s^2+(\frac{\sqrt{3}}{2}w)^2}]=\frac{2}{\sqrt{3}w}e^{-\frac{1}{2}wt}\cdot\sin\frac{\sqrt{3}}{2}wt$$

**3. (1) 單位步階函數(unit step function)定義為**

$$u(t-a)=\begin{cases}1, & t>a\\ 0, & t<a\end{cases}$$

**其中 *a* 為常數，試求**$u(t-5)$**之拉普拉斯轉換。 (5%)**

**(2) 試以單位步階函數之組合來表示下圖** $f(t)$ **之函數。 (5%)**




**(3) 試求** $f(t)$ **之拉普拉斯轉換，即** $F(s)=?$ **(5%)**

(1) $F(s)=\mathcal{L}[u(t-5)]=\int_0^\infty u(t-5)\,e^{-st}\,dt=\int_5^\infty e^{-st}\,dt=\frac{1}{s}e^{-5s}$

(2) $f(t)=2[u(t)-u(t-1)]+[u(t-1)-u(t-2)]=2u(t)-u(t-1)-u(t-2)$

(3) $\mathcal{L}[f(t)]=\frac{1}{s}(2e^{-0\cdot s}-e^{-1\cdot s}-e^{-2\cdot s})=\frac{1}{s}(2-e^{-s}-e^{-2s})$

**4. 試以拉普拉斯轉換法求解下述方程式。 (20%)**

**(1)** $y'(t) = \cos t + \int_0^t y(\tau)\cos(t-\tau)\,d\tau$ 且 $y(0) = 1$

**(2)** $y''(t) + 3y'(t) + 2y(t) = 3\delta(x-2)$ 且 $y'(0) = y(0) = 0$

(1) $\mathcal{L}[y(t)] = Y(s)$

又 $\mathcal{L}[\int_0^t y(\tau)\cos(t-\tau)\,d\tau] = \mathcal{L}[y(t) * \cos t] = Y(s)\cdot\dfrac{s}{s^2+1}$

$\therefore \mathcal{L}[y'(t)] = \mathcal{L}[\cos t + \int_0^t y(\tau)\cos(t-\tau)\,d\tau]$

$$\Rightarrow sY(s) - y(0) = \frac{s}{s^2+1} + Y(s)\cdot\frac{s}{s^2+1}$$

$$\Rightarrow (s - \frac{s}{s^2+1})Y(s) = \frac{s}{s^2+1} + 1$$

$$\Rightarrow \frac{s^3}{s^2+1}Y(s) = \frac{s^2+s+1}{s^2+1}$$

$$\Rightarrow Y(s) = \frac{s^2+s+1}{s^3}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}[\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3}] = 1 + t + \frac{t^2}{2}$$

(2) $\mathcal{L}[y''(t) + 3y'(t) + 2y(t)] = \mathcal{L}[3\delta(x-2)]$

$$\Rightarrow [s^2Y(s) - sy(0) - y'(0)] + 3[sY(s) - y(0)] + 2Y(s) = 3e^{-2s}$$

$$\Rightarrow (s^2+3s+2)Y(s) = 3e^{-2s}$$

$$\Rightarrow Y(s) = \frac{3e^{-2s}}{s^2+3s+2}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}[\frac{3e^{-2s}}{(s+1)(s+2)}] = 3\mathcal{L}^{-1}[e^{-2s}(\frac{1}{s+1} - \frac{1}{s+2})]$$

$$= 3[e^{-(t-2)} - e^{-2(t-2)}]\cdot u(t-2)$$

**5. 試求解下述聯立微分方程組 (10%)**

$$\begin{cases} \dfrac{dy_1}{dt} = -y_1 - y_2 \\ \dfrac{dy_2}{dt} = y_1 - y_2 \end{cases}$$ 且 $y_1(0) = 0$ 與 $y_2(0) = 1$

$\mathcal{L}[y_1(t)] = Y_1(s)$ 與 $\mathcal{L}[y_2(t)] = Y_2(s)$

將聯立微分方程組 $\begin{cases} \dfrac{dy_1}{dt} = -y_1 - y_2 \\ \dfrac{dy_2}{dt} = y_1 - y_2 \end{cases}$ 做拉普拉斯轉換後可得

$$\begin{cases} sY_1(s) - y_1(0) = -Y_1(s) - Y_2(s) \\ sY_2(s) - y_2(0) = Y_1(s) - Y_2(s) \end{cases}$$

$$\Rightarrow \begin{cases} (s+1)Y_1(s) + Y_2(s) = 0 \\ -Y_1(s) + (s+1)Y_2(s) = 1 \end{cases}$$

$$\Rightarrow Y_1(s) = -\frac{1}{(s+1)^2+1}, \quad Y_2(s) = \frac{s+1}{(s+1)^2+1}$$

$$\Rightarrow y_1(t) = \mathcal{L}^{-1}[Y_1(s)] = -e^{-t}\sin t, \quad y_2(t) = \mathcal{L}^{-1}[Y_2(s)] = e^{-t}\cos t$$

**6.** 若 $f(t) = 2\sin 2t$ ，$g(t) = \cos 2t$ ，試由求 $f(t) * g(t)$ 。**(10%)**

由 $\mathcal{L}[f(t)] = \mathcal{L}[2\sin 2t] = \dfrac{4}{s^2+4} = F(s)$

$$\mathcal{L}[g(t)] = \mathcal{L}[\cos 2t] = \frac{s}{s^2+4} = G(s)$$

$$\Rightarrow \mathcal{L}[f(t) * g(t)] = F(s) \cdot G(s) = \frac{4}{s^2+4} \cdot \frac{s}{s^2+4} = \frac{4s}{(s^2+4)^2}$$

$$\Rightarrow f(t) * g(t) = \mathcal{L}^{-1}[\frac{4s}{(s^2+4)^2}] = \mathcal{L}^{-1}[-\frac{d}{ds}(\frac{2}{s^2+4})] = t\sin 2t$$