

系級：_____ 學號：_____ 姓名：_____

1. 已知 1 階微分方程式 $y'(x) + y(x) = y^4(x)$ 具有 $u(x, y) = e^{ax} y^b$ 之積分因子

- (1) 試求此積分因子中 a 與 b 之值。(8%)
- (2) 試求此微分方程式之解 $y(x) = ?$ (6%)

2. 給一 Clairauts 微分方程式 $(y')^2 - xy' + y = 0$ ，試求此微分方程的通解(general solution)與奇解(singular solution)。(10%)

3. 已知微分方程式為 $(x^2 - 1)\frac{dy}{dx} + 2y = (x + 1)^2$

- (1) 此微分方程式為線性或非線性? (3%) 並以一階線性法求解。(8%)
(若為線性，直接求解；若非線性，使用變數變換法轉成線性，再求解)
- (2) 此微分方程式為正合(exact)或非正合? (5%) 並以正合法求解。(8%)
(若正合，直接求解；若非正合，先求出積分因子，再求解)

4. 已知微分方程式為 $\frac{dy}{dx} = x^3(y - x)^2 + x^{-1}y, x > 0$

- (1) 此為何種類型之微分方程式? (3%) 為線性或非線性? (3%)
- (2) 試以觀察法得一特解 S 。 (5%)
- (3) 試由變數變換($y = W + S$)將上述 ODE 轉換為以 W 表示之微分方程。 (6%)
請問：轉換後為何種類型微分方程式。 (3%)
- (4) 試求 $W = ?$ 並寫出最後解的表示式($y = W + S$)。 (8%)

5. 試解下列各微分方程

$$(1) [e^{(x+y)} + ye^y] + xe^y \frac{dy}{dx} = 0 \quad (8\%)$$

$$(2) (2y + e^y + 6x^2) \frac{dy}{dx} + 4 + 12xy = 0 \quad (8\%)$$

$$(3) x \frac{dy}{dx} - 2y = x^3 \cos 4x \quad (8\%)$$

<參考解答>

1. 已知 1 階微分方程式 $y'(x) + y(x) = y^4(x)$ 具有 $u(x, y) = e^{ax}y^b$ 之積分因子

- (1) 試求此積分因子中 a 與 b 之值。(8%)
- (2) 試求此微分方程式之解 $y(x) = ?$ (6%)

(1) $\because u(x, y) = e^{ax}y^b$ 為積分因子

\therefore 將微分方程式同乘積分因子後可得

$$e^{ax}y^b y'(x) + e^{ax}y^b y(x) = e^{ax}y^b y^4(x) \Rightarrow e^{ax}y^b(y - y^4)dx + e^{ax}y^b dy = 0$$

$$\text{令 } M = e^{ax}y^b(y - y^4) \Rightarrow \frac{\partial M}{\partial y} = e^{ax}[(b+1)y^b - (b+4)y^{b+3}]$$

$$N = e^{ax}y^b \Rightarrow \frac{\partial N}{\partial x} = ae^{ax}y^b$$

$$\text{又此為正合 ODE, 故可知 } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

比較係數後可得 $b+4=0 \Rightarrow b=-4$

$$a=b+1 \Rightarrow a=-3$$

(2) 將係數代入後可得 $e^{-3x}(y^{-3}-1)dx + e^{-3x}y^{-4}dy = 0$

此為正合 ODE, 故可得

$$\frac{\partial \phi}{\partial x} = M = e^{-3x}(y^{-3}-1) \Rightarrow \phi = -\frac{1}{3}e^{-3x}(y^{-3}-1) + f(y)$$

$$\frac{\partial \phi}{\partial y} = N = e^{-3x}y^{-4} \Rightarrow \phi = -\frac{1}{3}e^{-3x}y^{-3} + g(x)$$

$$\text{比較後可得 } \phi(x, y) = -\frac{1}{3}e^{-3x}(y^{-3}-1) = c$$

2. 給一 Clairauts 微分方程式 $(y')^2 - xy' + y = 0$, 試求此微分方程的通解(general solution)與奇解(singular solution)。(10%)

$$(y')^2 - xy' + y = 0 \Rightarrow y = xy' - (y')^2$$

令 $u = y'$ 代回 ODE 可得 $y = xu - u^2$

同時將兩邊對 x 微分: $y' = u + xu' - 2uu' \Rightarrow u = u + xu' - 2uu'$

$$\Rightarrow u'(x-2u) = 0$$

$$\Rightarrow u' = 0 \text{ 或 } u = \frac{x}{2}$$

當 $u' = 0 \Rightarrow u = c \Rightarrow y = cx - c^2$ (通解)

當 $u = \frac{x}{2} \Rightarrow y = \frac{x^2}{2} - \frac{x^2}{4} = \frac{x^2}{4}$ (奇解)

3. 已知微分方程式為 $(x^2 - 1)\frac{dy}{dx} + 2y = (x+1)^2$

(1) 此微分方程式為線性或非線性? (3%) 並以一階線性法求解。(8%)

(若為線性，直接求解；若非線性，使用變數變換法轉成線性，再求解)

(2) 此微分方程式為正合(exact)或非正合? (5%) 並以正合法求解。(8%)

(若正合，直接求解；若非正合，先求出積分因子，再求解)

$$(1) (x^2 - 1)\frac{dy}{dx} + 2y = (x+1)^2 \Rightarrow \frac{dy}{dx} + \frac{2}{(x^2 - 1)}y = \frac{x+1}{x-1}$$

此為一階線性 ODE

$$\text{積分因子為 } \mu = e^{\int p(x)dx} = e^{\int \frac{2}{(x^2-1)}dx} = e^{\int (\frac{1}{x-1} - \frac{1}{x+1})dx} = \frac{x-1}{x+1}$$

$$\text{同乘積分因子後可得 } \frac{x-1}{x+1} \frac{dy}{dx} + \frac{2}{(x+1)^2} y = 1$$

$$\Rightarrow \frac{d}{dx} \left(\frac{x-1}{x+1} y \right) = 1$$

$$\Rightarrow \int d \left(\frac{x-1}{x+1} y \right) = \int dx$$

$$\Rightarrow \frac{x-1}{x+1} y = x + c$$

$$\Rightarrow y = (x+c) \frac{x+1}{x-1}$$

$$(2) (x^2 - 1)\frac{dy}{dx} + 2y = (x+1)^2 \Rightarrow [(x+1)^2 - 2y]dx - (x^2 - 1)dy = 0$$

$$\text{令 } M = (x+1)^2 - 2y \Rightarrow \frac{\partial M}{\partial y} = -2$$

$$N = -(x^2 - 1) \Rightarrow \frac{\partial N}{\partial x} = -2x$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \text{ 此為非正合微分方程}$$

$$\therefore \text{由 } \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = -\frac{1}{x^2 - 1} (-2 + 2x) = -\frac{2}{x+1} \text{ 可知積分因子 } \mu = \mu(x)$$

$$\text{故可得 } \int \frac{1}{\mu} d\mu = \int \frac{1}{N} \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) dx = -\int \frac{2}{x+1} dx$$

$$\Rightarrow \ln \mu = -2 \ln(x+1)$$

$$\Rightarrow \mu = \frac{1}{(x+1)^2}$$

$$\text{同乘積分因子後可得 } [1 - \frac{2y}{(x+1)^2}]dx - \frac{x-1}{x+1} dy = 0 \text{ 為正合 ODE}$$

$$\begin{aligned}\therefore M &= \frac{\partial \phi}{\partial x} = 1 - \frac{2y}{(x+1)^2} \Rightarrow \phi = x - \frac{2y}{x+1} + f(y) \\ N &= \frac{\partial \phi}{\partial y} = \frac{x-1}{x+1} \Rightarrow \phi = \frac{x-1}{x+1} y + g(x) = 1 - \frac{2y}{x+1} + g(x) \\ \therefore \text{解為 } \phi(x, y) &= c \Rightarrow x - \frac{2y}{x+1} + 1 = c \\ &\Rightarrow x - c = \frac{x-1}{x+1} y \\ &\Rightarrow y = (x-c) \frac{x+1}{x-1}\end{aligned}$$

4. 已知微分方程式為 $\frac{dy}{dx} = x^3(y-x)^2 + x^{-1}y, x > 0$

- (1) 此為何種類型之微分方程式? (3%) 為線性或非線性? (3%)
- (2) 試以觀察法得一特解 S 。 (5%)
- (3) 試由變數變換 ($y = W + S$) 將上述 ODE 轉換為以 W 表示之微分方程。 (6%)
請問：轉換後為何種類型微分方程式。 (3%)
- (4) 試求 $W = ?$ 並寫出最後解的表示式 ($y = W + S$)。 (8%)

$$(1) \frac{dy}{dx} = x^3(y-x)^2 + x^{-1}y \Rightarrow \frac{dy}{dx} = x^3y^2 + \left(\frac{1}{x} - 2x^4\right)y + x^5$$

此為 Riccati ODE，為非線性微分方程式

- (2) 由觀察可得一解 $S = x$
- (3) 令 $y = W + S = W + x$ 代入 ODE 可得

$$W' + 1 = x^3(W+x)^2 + \left(\frac{1}{x} - 2x^4\right)(W+x) + x^5$$

$$\Rightarrow W' - \frac{1}{x}W = x^3W^2$$

\therefore 可知此為 Bernoulli ODE

$$(4) W' - \frac{1}{x}W = x^3W^2 \Rightarrow W^{-2}W' - \frac{1}{x}W^{-1} = x^3$$

令 $u = W^{-1} \Rightarrow u' = -W^{-2}W'$ 代回 ODE 可得

$$u' + \frac{1}{x}u = -x^3 \quad \longrightarrow \text{此為一階線性 ODE}$$

$$\therefore \text{積分因子 } \mu = e^{\int p(x)dx} = e^{\int \frac{1}{x}dx} = x$$

將 ODE 同乘積分因子後可得 $xu' + u = -x^4$

$$\Rightarrow \frac{d}{dx}(xu) = -x^4$$

$$\begin{aligned}
&\Rightarrow xu = -\frac{1}{5}x^5 + c \\
&\Rightarrow u = -\frac{1}{5}x^4 + \frac{c}{x} \\
&\Rightarrow \frac{1}{W} = -\frac{1}{5}x^4 + \frac{c}{x} \\
&\Rightarrow W = \frac{x}{-\frac{1}{5}x^5 + c}
\end{aligned}$$

$$y = W + S = \frac{x}{-\frac{1}{5}x^5 + c} + x$$

5. 試解下列各微分方程

$$(1) [e^{(x+y)} + ye^y] + xe^y \frac{dy}{dx} = 0 \quad (8\%)$$

$$(2) (2y + e^y + 6x^2) \frac{dy}{dx} + 4 + 12xy = 0 \quad (8\%)$$

$$(3) x \frac{dy}{dx} - 2y = x^3 \cos 4x \quad (8\%)$$

$$\begin{aligned}
(1) [e^{(x+y)} + ye^y] + xe^y \frac{dy}{dx} = 0 &\Rightarrow e^x dx + x dy + y dx = 0 \\
&\Rightarrow e^x dx + d(xy) = 0 \\
&\Rightarrow \int e^x dx + \int d(xy) = 0 \\
&\Rightarrow e^x + xy = c \\
&\Rightarrow y = \frac{c}{x} - \frac{1}{x} e^x
\end{aligned}$$

$$\begin{aligned}
(2) (2y + e^y + 6x^2) \frac{dy}{dx} + 4 + 12xy &= 0 \\
&\Rightarrow (2y + e^y + 6x^2) dy + (4 + 12xy) dx = 0 \\
&\Rightarrow (2y + e^y) dy + 4 dx + 6x(x dy + 2y dx) = 0 \\
&\Rightarrow (2y + e^y) dy + 4 dx + 6x \cdot \frac{1}{x} d(x^2 y) = 0 \\
&\Rightarrow \int (2y + e^y) dy + \int 4 dx + \int 6d(x^2 y) = 0 \\
&\Rightarrow y^2 + e^y + 4x + 6x^2 y = c
\end{aligned}$$

$$(3) \quad x \frac{dy}{dx} - 2y = x^3 \cos 4x$$
$$\Rightarrow y' - \frac{2}{x}y = x^2 \cos 4x \quad \text{一階線性 ODE}$$

積分因子為 $\mu = e^{\int p(x)dx} = e^{-\int \frac{2}{x}dx} = \frac{1}{x^2}$

同乘積分因子可得 $\frac{1}{x^2} y' - \frac{2}{x^3} y = \cos 4x$

$$\Rightarrow \frac{d}{dx} \left(\frac{1}{x^2} y \right) = \cos 4x$$

$$\Rightarrow \int d \left(\frac{1}{x^2} y \right) = \int \cos 4x dx$$

$$\Rightarrow \frac{1}{x^2} y = \frac{1}{4} \sin 4x + c$$

$$\Rightarrow y = \frac{1}{4} x^2 \sin 4x + cx^2$$