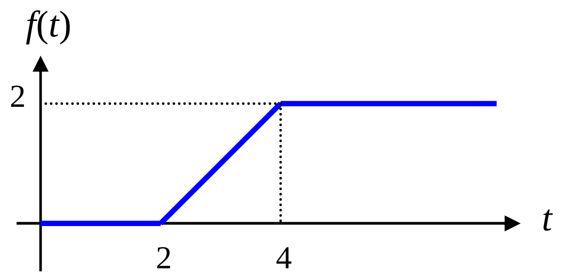
國立臺灣海洋大學河海工程學系 2017 工程數學(二) 期末考參考解答

系級：______________　　學號：____________　　姓名：___________

1. (1) 試以 unit step function 之組合來表示下圖 $f(t)$ 之函數。 (5%)



(2) 試求 $f(t)$ 之拉普拉斯轉換，即 $F(s)=?$ (5%)

(Hint: unit step function: $u(t-a)=\begin{cases}1, & t>a\\ 0, & t<a\end{cases}$ 其中 a 為常數)

2. 試求下列 $f(t)$ 的拉普拉斯轉換為何? (15%)

(1) $f(t)=5$　(2) $f(t)=t^2e^{-2t}$　(3) $f(t)=\sin(5t+\frac{\pi}{4})$

3. 試求下列 $F(s)$ 的拉普拉斯逆轉換為何? (15%)

(1) $F(s)=5$　(2) $F(s)=\dfrac{1}{s^2+s^4}$　(3) $F(s)=\dfrac{s+4}{s^2+4s+20}$

4. 已知 $f(t)=\begin{cases}1, & 0<x<1\\ -1, & 1<x<2\end{cases}$ 又 $f(t)=f(t+2)$，試問此函數經由拉普拉斯轉換後結果為何? (10%)

5. 若 $t\in[0,\infty)$，試問 $t^2*t^2*t^2=?$ (* 表示摺積運算元) (10%)

6. 試以拉普拉斯轉換法求解下述方程式。 (30%)

(1) $y'(t)=1-\sin t-\int_0^t y(\tau)d\tau$ 且 $y(0)=0$

(2) $y(t)=1-\sin t+\int_0^t y(\tau)\sin(t-\tau)d\tau$

(3) $y''(t)+3y'(t)+2y(t)=\delta(t-2)$ 且 $y'(0)=y(0)=0$

7. 試求解下述聯立微分方程組 (10%)

$$\begin{cases}x''(t)-4x'(t)+8y'(t)+4y(t)=4\\ 2y'(t)-2x'(t)+y(t)=0\end{cases}$$ 且 $x(0)=x'(0)=y(0)=1$

拉普拉斯轉換： $F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)\, e^{-st}\, dt$

第一平移定理: $\mathcal{L}[e^{at} f(t)] = F(s-a)$

第二平移定理: $\mathcal{L}[f(t-a)u(t-a)] = e^{-as} F(s)$

尺度變換: $\mathcal{L}[f(at)] = \frac{1}{a} F(\frac{s}{a})$

微分函數的拉普拉斯轉換: $\mathcal{L}[f'(t)] = sF(s) - f(0)$

$$\mathcal{L}[f''(t)] = s^2 F(s) - sf(0) - f'(0)$$

積分函數的拉普拉斯轉換: $\mathcal{L}[\int_0^t f(x)\, dx] = \frac{F(s)}{s}$

$$\mathcal{L}[\int_0^t \int_0^t f(x)\, dx\, dx] = \frac{F(s)}{s^2}$$

拉普拉斯轉換的微分: $\mathcal{L}[tf(t)] = (-1)\frac{d}{ds} F(s)$

$$\mathcal{L}[t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2} F(s)$$

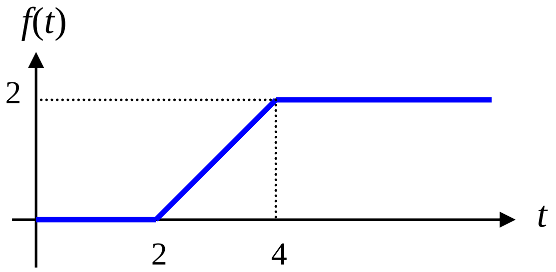
拉普拉斯轉換的積分: $\mathcal{L}[\frac{f(t)}{t}] = \int_s^\infty f(\tau)\, d\tau$

$$\mathcal{L}[\frac{f(t)}{t^2}] = \int_s^\infty \int_\gamma^\infty f(\tau)\, d\tau\, d\gamma$$

摺積: $f(t) * g(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$

參考解答：

1. (1) 試以 unit step function 之組合，寫出下圖 $f(t)$ 之函數式。 (5%)



(2) 試求 $f(t)$ 之拉普拉斯轉換，即 $F(s)=?$ (5%)

(1) $f(t)=(t-2)[u(t-2)-u(t-4)]+2u(t-4)$

$=(t-2)u(t-2)-(t-2)u(t-4)+2u(t-4)$

$=(t-2)u(t-2)-(t-4)u(t-4)$

(2) $\mathcal{L}[f(t)]=\frac{e^{-2s}}{s^2}-\frac{e^{-4s}}{s^2}$

2. 試求下列 $f(t)$ 的拉普拉斯轉換為何? (15%)

(1) $f(t)=5$ **(2)** $f(t)=t^2e^{-2t}$ **(3)** $f(t)=\sin(5t+\frac{\pi}{4})$

(1) $f(t)=5 \Rightarrow \mathcal{L}[f(t)]=\frac{5}{s}$

(2) $f(t)=t^2e^{-2t} \Rightarrow \mathcal{L}[f(t)]=\frac{2}{(s+2)^3}$

(3) $f(t)=\sin(5t+\frac{\pi}{4})=\sin 5t\cos\frac{\pi}{4}+\cos 5t\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}\sin 5t+\frac{1}{\sqrt{2}}\cos 5t$

$\Rightarrow \mathcal{L}[f(t)]=\frac{1}{\sqrt{2}}\frac{5}{s^2+25}+\frac{1}{\sqrt{2}}\frac{s}{s^2+25}=\frac{1}{\sqrt{2}}\frac{s+5}{s^2+25}$

3. 試求下列 $F(s)$ 的拉普拉斯逆轉換為何? (15%)

(1) $F(s)=5$ **(2)** $F(s)=\frac{1}{s^2+s^4}$ **(3)** $F(s)=\frac{s+4}{s^2+4s+20}$

(1) $F(s)=5 \Rightarrow \mathcal{L}^{-1}[F(s)]=5\delta(t)$

(2) $F(s)=\frac{1}{s^2+s^4}=\frac{1}{s^2(s^2+1)}=\frac{1}{s^2}-\frac{1}{s^2+1} \Rightarrow \mathcal{L}^{-1}[F(s)]=t-\sin t$

(3) $F(s)=\frac{s+4}{s^2+4s+20}=\frac{(s+2)+2}{(s+2)^2+4^2}$

$\Rightarrow \mathcal{L}^{-1}[F(s)]=\mathcal{L}^{-1}[\frac{(s+2)+2}{(s+2)^2+4^2}]=e^{-2t}\mathcal{L}^{-1}[\frac{s+2}{s^2+4^2}]=e^{-2t}(\cos 4t+\frac{1}{2}\sin 4t)$

4. 已知 $f(t)=\begin{cases}1, & 0<x<1\\ -1, & 1<x<2\end{cases}$ 又 $f(t)=f(t+2)$，試問此函數經由拉普拉斯轉換為何? **(10%)**

可知此為週期函數且週期$T=2$

$$
\begin{aligned}
\mathcal{L}[f(t)] &= \frac{1}{1-e^{-sT}}\int_0^T f(\tau)e^{-s\tau}d\tau \\
&= \frac{1}{1-e^{-2s}}\left[\int_0^1 1\cdot e^{-s\tau}d\tau+\int_1^2(-1)e^{-s\tau}d\tau\right] \\
\therefore \quad &= \frac{1}{1-e^{-2s}}\left[-\frac{1}{s}e^{-s\tau}\Big|_0^1+\frac{1}{s}e^{-s\tau}\Big|_1^2\right] \\
&= \frac{1}{1-e^{-2s}}\left[\frac{1}{s}e^{-2s}-\frac{2}{s}e^{-s}+\frac{1}{s}\right] \\
&= \frac{(1-e^{-s})^2}{s(1-e^{-2s})}=\frac{1-e^{-s}}{s(1+e^{-s})}
\end{aligned}
$$

5. 若$t\in[0,\infty)$，試問 $t^2 * t^2 * t^2 = ?$ **(10%)**

$$\mathcal{L}[t^2]=\frac{2}{s^3}$$

$$t^2*t^2*t^2=\mathcal{L}^{-1}[\mathcal{L}[t^2*t^2*t^2]]=\mathcal{L}^{-1}[\mathcal{L}[t^2]\cdot\mathcal{L}[t^2*t^2]]=\mathcal{L}^{-1}[\mathcal{L}[t^2]\cdot\mathcal{L}[t^2]\cdot\mathcal{L}[t^2]]$$

$$=\mathcal{L}^{-1}\left[\frac{2}{s^3}\cdot\frac{2}{s^3}\cdot\frac{2}{s^3}\right]=\mathcal{L}^{-1}\left[\frac{8}{s^9}\right]=\frac{t^8}{7!}$$

6. 試以拉普拉斯轉換法求解下述方程式。 **(30%)**

(1) $y'(t)=1-\sin t-\int_0^t y(\tau)d\tau$ 且 $y(0)=0$

(2) $y(t)=1-\sin t+\int_0^t y(\tau)\sin(t-\tau)d\tau$

(3) $y''(t)+3y'(t)+2y(t)=\delta(t-2)$ 且 $y'(0)=y(0)=0$

(1) $\mathcal{L}[y'(t)]=\mathcal{L}[1-\sin t-\int_0^t y(\tau)d\tau]$

$$\Rightarrow sY(s)-y(0)=\frac{1}{s}-\frac{1}{s^2+1}-\frac{1}{s}Y(s)$$

$$\Rightarrow \frac{s^2+1}{s}Y(s)=\frac{1}{s}-\frac{1}{s^2+1}$$

$$\Rightarrow Y(s)=\frac{1}{s^2+1}-\frac{s}{(s^2+1)^2}$$

$$\Rightarrow y(t)=\mathcal{L}^{-1}[Y(s)]=\sin t-\frac{1}{2}t\sin t$$

(2) $\mathcal{L}[y(t)] = \mathcal{L}[1 - \sin t - \int_0^t y(\tau)\sin(t-\tau)\,d\tau]$

$$\Rightarrow Y(s) = \frac{1}{s} - \frac{1}{s^2+1} + \frac{1}{s^2+1}Y(s)$$

$$\Rightarrow \frac{s^2}{s^2+1}Y(s) = \frac{1}{s} - \frac{1}{s^2+1}$$

$$\Rightarrow Y(s) = \frac{s^2+1}{s^3} - \frac{1}{s^2}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}[Y(s)] = 1 - t + \frac{1}{2}t^2$$

(3) $\mathcal{L}[y''(t) + 3y'(t) + 2] = \mathcal{L}[\delta(t-2)]$

$\Rightarrow [s^2Y(s) - sy(0) - y'(0)] + 3[sY(s) - y(0)] + 2Y(s) = e^{-2s}$

又 $y'(0) = y(0) = 0$

$\therefore s^2Y(s) + 3sY(s) + 2Y(s) = e^{-2s}$

$$\Rightarrow Y(s) = \frac{e^{-2s}}{s^2+3s+2} = e^{-2s}[\frac{1}{s+1} - \frac{1}{s+2}]$$

$\therefore y(t) = \mathcal{L}^{-1}[Y(s)] = [e^{-(t-2)} - e^{-2(t-2)}]\,u(t-2)$

7. 試求解下述聯立微分方程組 (10%)

$$\begin{cases} x''(t) - 4x'(t) + 8y'(t) + 4y(t) = 4 \\ 2y'(t) - 2x'(t) + y(t) = 0 \end{cases} \quad \text{且} \quad x(0) = x'(0) = y(0) = 1$$

由拉普拉斯轉換可得

$$\begin{cases} s^2X(s) - sx(0) - x'(0) - 4[sX(s) - x(0)] + 8[sY(s) - y(0)] + 4Y(s) = \dfrac{4}{s} \\ 2[sY(s) - y(0)] - 2[sX(s) - x(0)] + Y(s) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} (s^2 - 4s)X(s) + (8s+4)Y(s) = s + 5 + \dfrac{4}{s} & \cdots\cdots (1) \\ -2sX(s) + (2s+1)Y(s) = 0 & \cdots\cdots (2) \end{cases}$$

由 (2) 可知 $X(s) = \dfrac{2s+1}{2s}Y(s)$ 代回 (1) 可得

$$\frac{(s^2-4s)(2s+1)}{2s}Y(s) + (8s+4)Y(s) = s + 5 + \frac{4}{s}$$

$$\Rightarrow \frac{2s^3 + 9s^2 + 4s}{2s}Y(s) = \frac{s^2+5s+4}{s}$$

$$\Rightarrow Y(s) = \frac{2s^2 + 10s + 8}{2s^3 + 9s^2 + 4s}$$

$$\therefore \quad X(s) = \frac{2s^3 + 11s^2 + 13s + 4}{2s^4 + 9s^3 + 4s^2}$$

又 $Y(s)=\dfrac{2s^2+10s+8}{2s^3+9s^2+4s}=\dfrac{2s^2+10s+8}{s(s+4)(2s+1)}=\dfrac{A}{s}+\dfrac{B}{s+4}+\dfrac{C}{2s+1}$

$\Rightarrow A(s+4)(2s+1)+Bs(2s+1)+Cs(s+4)=2s^2+10s+8$

當 $s=0 \quad \Rightarrow A=2$

當 $s=-4 \quad \Rightarrow B=0$

當 $s=-\dfrac{1}{2} \quad \Rightarrow C=-2$

$\therefore Y(s)=\dfrac{2}{s}-\dfrac{2}{2s+1}=\dfrac{2}{s}-\dfrac{1}{s+\frac{1}{2}} \quad \Rightarrow y(t)=\mathcal{L}^{-1}[Y(s)]=2-e^{-\frac{1}{2}t}$

又 $X(s)=\dfrac{2s^3+11s^2+13s+4}{2s^4+9s^3+4s^2}=\dfrac{As+B}{s^2}+\dfrac{C}{s+4}+\dfrac{D}{2s+1}$

$\Rightarrow (As+B)(s+4)(2s+1)+Cs^2(2s+1)+Ds^2(s+4)=2s^3+11s^2+13s+4$

當 $s=0 \quad \Rightarrow B=1$

當 $s=-4 \quad \Rightarrow C=0$

當 $s=-\dfrac{1}{2} \quad \Rightarrow D=0$

比較 s^3 項可得 $A=1$

$\therefore X(s)=\dfrac{2s^3+11s^2+13s+4}{2s^4+9s^3+4s^2}=\dfrac{s+1}{s^2}=\dfrac{1}{s}+\dfrac{1}{s^2} \quad \Rightarrow x(t)=\mathcal{L}^{-1}[X(s)]=1+t$