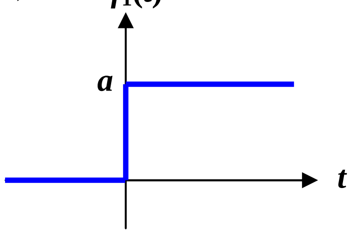
國立臺灣海洋大學河海工程學系 2016 工程數學(二) 期末考參考解答

系級：______________ 學號：______________ 姓名：______________

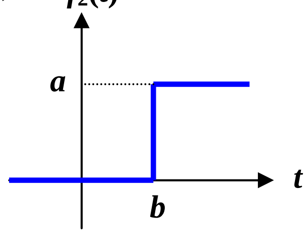
1. 試求下列各圖函數的拉普拉斯轉換為何? (15%)

(1) $f_1(t)$



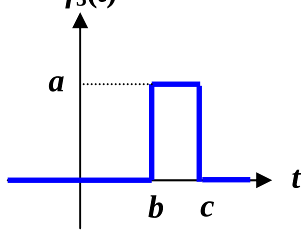
$\mathcal{L}[f_1(t)] = ?$

(2) $f_2(t)$



$\mathcal{L}[f_2(t)] = ?$

(3) $f_3(t)$



$\mathcal{L}[f_3(t)] = ?$

2. 試求下列各函數的拉普拉斯轉換為何? (15%)

(1) $\delta(t-3)$ **(2)** $\cos(t-\frac{\pi}{4})$ **(3)** $t\sin(2t)$

3. 試求下列 $F(s)$ 的拉普拉斯逆轉換。 (15%)

(1) $F(s)=\dfrac{1}{(s-2)^4}$ **(2)** $F(s)=\dfrac{s^2}{s^2-9}$ **(3)** $F(s)=\dfrac{e^{-3s}}{s^2+2s+2}$

4. 已知 $\mathcal{L}[f(t)]=F(s)$, $\mathcal{L}[g(t)]=G(s)$ 且 $h(t)$ 為 $f(t)$ 與 $g(t)$ 的摺積(Convolution)

即 $h(t)=f(t)*g(t)=\int_0^t f(\tau)g(t-\tau)\,d\tau$，並且由 $h(t)$ 拉普拉斯轉換可得

$\mathcal{L}[h(t)]=H(s)=F(s)G(s)$，若已知 $H(s)=\dfrac{s^2}{(s^2+a^2)^2}$，試求 $h(t)=?$ **(15%)**

5. 試以 Laplace 轉換法求解第二類 Volterra 積分方程式

(1) $y(t)-\int_0^t y(\tau)\sin(t-\tau)\,d\tau=t$ **(10%)**

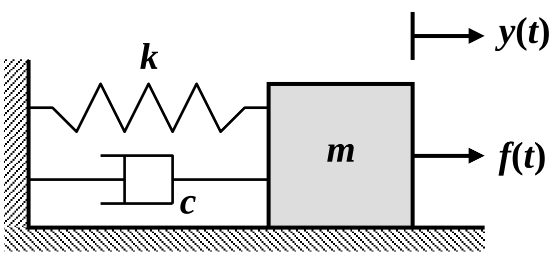
(2) $y(t)-\int_0^t (1+\tau)y(t-\tau)\,d\tau=1-\sinh t$ **(10%)** **(104 台大土木)**

6. 考慮一單自由度振動系統如圖所示，其數學模式可寫成 $my''+cy'+ky=f(t)$，並且此系統一開始為靜止狀態，即 $y(0)=y'(0)=0$。當外力 $f(t)=\delta(t)$ 時，可得解為 $y(t)=e^{-t}-e^{-2t}$，試問：

(1) $m=?$ $c=?$ 與 $k=?$ **(10%)**

(2) 若外力改給 $f(t)=1$，$0<t<1$，其餘皆為零時，此時 $y(t)=?$ **(10%)**

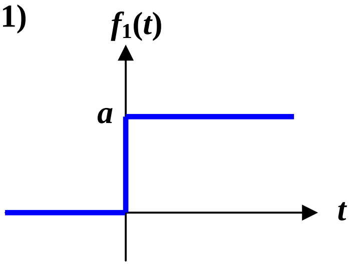
(105 交大土木丁組)



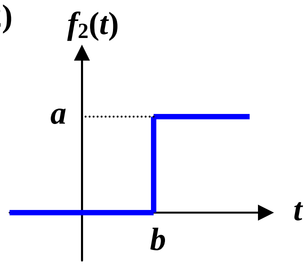
參考解答：

1. 試求下列各圖函數的拉普拉斯轉換為何? (15%)

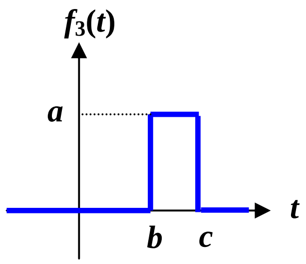
(1)



(2)



(3)



$\mathcal{L}[f_1(t)] = \dfrac{a}{s}$ $\mathcal{L}[f_2(t)] = \dfrac{a}{s}e^{-bs}$ $\mathcal{L}[f_3(t)] = \dfrac{a}{s}(e^{-bs} - e^{-cs})$

2. 試求下列各函數的拉普拉斯轉換為何? (15%)

(1) $\delta(t-3)$ **(2)** $\cos(t-\dfrac{\pi}{4})$ **(3)** $t\sin(2t)$

(1) $\mathcal{L}[\delta(t-3)] = e^{-3s}$

(2) $\mathcal{L}[\cos(t-\dfrac{\pi}{4})] = \mathcal{L}[\cos\dfrac{\pi}{4}\cdot\cos t + \sin\dfrac{\pi}{4}\cdot\sin t]$

$$= \frac{1}{\sqrt{2}}\mathcal{L}[\cos t + \sin t]$$

$$= \frac{1}{\sqrt{2}}\frac{s+1}{s^2+1}$$

(3) $\mathcal{L}[t\sin(2t)] = -\dfrac{d}{ds}(\dfrac{2}{s^2+4}) = \dfrac{4s}{(s^2+4)^2}$

3. 試求下列 $F(s)$ 的拉普拉斯逆轉換。 (15%)

(1) $F(s) = \dfrac{1}{(s-2)^4}$ **(2)** $F(s) = \dfrac{s^2}{s^2-9}$ **(3)** $F(s) = \dfrac{e^{-3s}}{s^2+2s+2}$

(1) $\mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}[\dfrac{1}{(s-2)^4}] = e^{2t}\cdot\mathcal{L}^{-1}[\dfrac{1}{s^4}] = \dfrac{1}{6}t^3e^{2t}$

(2) $\mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}[\dfrac{s^2}{s^2-9}] = \mathcal{L}^{-1}[1+\dfrac{9}{s^2-9}] = \delta(t) + 3\sinh 3t$

(3) $\mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}[\dfrac{e^{-3s}}{s^2+2s+2}] = \mathcal{L}^{-1}[e^{-3s}\cdot\dfrac{1}{(s+1)^2+1}]$

$$\mathcal{L}^{-1}[\frac{1}{(s+1)^2+1}] = e^{-t}\cdot\mathcal{L}^{-1}[\frac{1}{s^2+1}] = e^{-t}\sin t$$

$$\therefore\ \mathcal{L}^{-1}[e^{-3s}\cdot\frac{1}{(s+1)^2+1}] = e^{-(t-3)}\sin(t-3)\cdot u(t-3)\quad (u(t):\text{unit step function})$$

4. 已知 $\mathcal{L}[f(t)]=F(s)$, $\mathcal{L}[g(t)]=G(s)$ 且 $h(t)$ 為 $f(t)$ 與 $g(t)$ 的摺積**(Convolution)**

即 $h(t)=f(t)*g(t)=\int_0^t f(\tau)g(t-\tau)\,d\tau$ ，並且由 $h(t)$ 拉普拉斯轉換可得

$\mathcal{L}[h(t)]=H(s)=F(s)G(s)$ ，若已知 $H(s)=\dfrac{s^2}{(s^2+a^2)^2}$ ，試求 $h(t)=?$ **(15%)**

$$H(s)=\frac{s}{s^2+a^2}\cdot\frac{s}{s^2+a^2}$$

$$\therefore \text{ 令 } F(s)=G(s)=\frac{s}{s^2+a^2}$$

$$\Rightarrow f(t)=g(t)=\mathcal{L}^{-1}[F(s)]=\mathcal{L}^{-1}[\frac{s}{s^2+a^2}]=\mathcal{L}^{-1}[\frac{s}{s^2+a^2}]=\cos at$$

$$\begin{aligned}
h(t)&=f(t)*g(t)\\
&=\int_0^t \cos a\tau\cdot\cos a(t-\tau)\,d\tau\\
&=\int_0^t \cos a\tau\cdot(\cos at\cos a\tau+\sin at\sin a\tau)\,d\tau\\
&=\int_0^t (\cos at\cos^2 a\tau+\frac{1}{2}\sin at\sin 2a\tau)\,d\tau\\
&=\int_0^t [\frac{1}{2}\cos at(1+\cos 2a\tau)+\frac{1}{2}\sin at\sin 2a\tau]\,d\tau\\
&=\frac{1}{2}[\cos at(\tau+\frac{1}{2a}\sin 2a\tau)-\frac{1}{2a}\sin at\cos 2a\tau]\Big|_0^t\\
&=\frac{1}{2}[\cos at(t+\frac{1}{2a}\sin 2at)-\frac{1}{2a}(\sin at\cos 2at-\sin at)]\\
&=\frac{1}{2}(t\cos at+\frac{1}{a}\sin at)\\
&=\frac{at\cos at+\sin at}{2a}
\end{aligned}$$

5. 試以 Laplace 轉換法求解第二類 Volterra 積分方程式

(1) $y(t)-\int_0^t y(\tau)\sin(t-\tau)\,d\tau = t$ **(10%)**

(2) $y(t)-\int_0^t (1+\tau)y(t-\tau)\,d\tau = 1-\sinh t$ **(10%)** **(104 台大土木)**

(1) $y(t)-\int_0^t y(\tau)\sin(t-\tau)\,d\tau = t$

$$\Rightarrow y(t)-y(t)*\sin(t)=t$$

$$\Rightarrow \mathcal{L}[y(t)-y(t)*\sin(t)]=\mathcal{L}[t]$$

$$\Rightarrow Y(s)-Y(s)\cdot\frac{1}{s^2+1}=\frac{1}{s^2}$$

$$\Rightarrow Y(s)\left[1-\frac{1}{s^2+1}\right]=\frac{1}{s^2}$$

$$\Rightarrow Y(s)=\frac{s^2+1}{s^2}\frac{1}{s^2}=\frac{1}{s^2}+\frac{1}{s^4}$$

$$\Rightarrow y(t)=\mathcal{L}^{-1}[Y(s)]=\mathcal{L}^{-1}\left[\frac{1}{s^2}+\frac{1}{s^4}\right]=t+\frac{1}{6}t^3$$

(2) $y(t)-\int_0^t (1+\tau)y(t-\tau)\,d\tau = 1-\sinh t$

$$\Rightarrow y(t)-y(t)*(1+t)=1-\sinh t$$

$$\Rightarrow \mathcal{L}[y(t)-y(t)*(1+t)]=\mathcal{L}[1-\sinh t]$$

$$\Rightarrow Y(s)-Y(s)\cdot\left[\frac{1}{s}+\frac{1}{s^2}\right]=\frac{1}{s}-\frac{1}{(s^2-1)}$$

$$\Rightarrow Y(s)\left[1-\frac{1}{s}-\frac{1}{s^2}\right]=\frac{s^2-s-1}{s(s^2-1)}$$

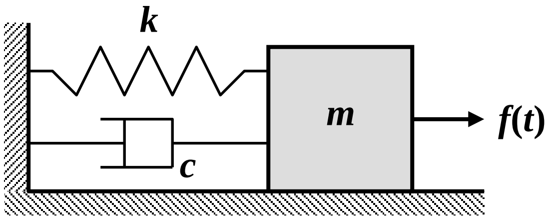
$$\Rightarrow Y(s)=\frac{s}{s^2-1}$$

$$\Rightarrow y(t)=\mathcal{L}^{-1}[Y(s)]=\mathcal{L}^{-1}\left[\frac{s}{s^2-1}\right]=\cosh t$$

6. 考慮一單自由度振動系統如圖所示，其數學模式可寫成 $my'' + cy' + ky = f(t)$，並且此系統一開始為靜止狀態，即 $y(0) = y'(0) = 0$。當外力 $f(t) = \delta(t)$ 時，可得解為 $y(t) = e^{-t} - e^{-2t}$，試問：

(1) $m = ?$，$c = ?$ 與 $k = ?$ **(10%)**

(2) 若外力改給 $f(t) = 1$，$0 < t < 1$，其餘皆為零時，此時 $y(t) = ?$ **(10%)**



(1) $my'' + cy' + ky = \delta(t)$

$\Rightarrow \mathcal{L}[my'' + cy' + ky] = \mathcal{L}[\delta(t)]$

$\Rightarrow m[s^2Y(s) - sy(0) - y'(0)] + c[sY(s) - y(0)] + kY(s) = 1$

$\Rightarrow Y(s)(ms^2 + cs + k) = 1$

$$\Rightarrow Y(s) = \frac{1}{ms^2 + cs + k}$$

$\because$ 此系統的解為 $y(t) = e^{-t} - e^{-2t}$

$$\therefore \mathcal{L}[y(t)] = \mathcal{L}[e^{-t} - e^{-2t}] \Rightarrow Y(s) = \frac{1}{s+1} - \frac{1}{s+2} = \frac{1}{s^2 + 3s + 2}$$

故可得 $m = 1$，$c = 3$ 與 $k = 2$

(2) $f(t) = 1$ 且 $0 < t < 1$，其餘皆為零

可表示成 $f(t) = u(t) - u(t-1)$

$\therefore$ 可知其數學模式為 $y'' + 3y' + 2y = u(t) - u(t-1)$

$\mathcal{L}[y'' + 3y' + 2y] = \mathcal{L}[u(t) - u(t-1)]$

$$\Rightarrow Y(s)(s^2 + 3s + 2) = \frac{1}{s}(1 - e^{-s})$$

$$\Rightarrow Y(s) = \frac{(1 - e^{-s})}{s(s^2 + 3s + 2)} = \frac{(1 - e^{-s})}{s(s+1)(s+2)}$$

$$\text{又} \quad \frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

通分並比較係數後可得 $A = \frac{1}{2}$，$B = -1$，$C = \frac{1}{2}$

$$\therefore \mathcal{L}[\frac{1}{s(s+1)(s+2)}] = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$$

$$\Rightarrow \mathcal{L}[\frac{(1 - e^{-s})}{s(s+1)(s+2)}] = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} - u(t-1)[\frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)}]$$

$$\therefore y(t) = \mathcal{L}^{-1}[Y(s)] = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} - u(t-1)[\frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)}]$$