國立臺灣海洋大學河海工程學系 2015 工程數學(一) 第二次隨堂測驗參考解答

系級：______________　　學號：____________　　姓名：___________

1. 試問此微分方程 $(x+y)dx+(x\cdot\ln x)dy=0$ 是否正合? 若是請求出其解；
   若不是，請問其積分因子為何? 並試求出其解。

令 $M=x+y \quad \Rightarrow \dfrac{\partial M}{\partial y}=1$

$N=x\cdot\ln x \quad \Rightarrow \dfrac{\partial N}{\partial x}=\ln x+1$

由判別式 $\dfrac{\partial M}{\partial y}\neq\dfrac{\partial N}{\partial x}$ 可知此非正合

由 $\dfrac{1}{N}(\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x})=-\dfrac{\ln x}{x\cdot\ln x}=-\dfrac{1}{x} \quad \Rightarrow \mu$ 是 $\mu(x)$

$\therefore \dfrac{1}{\mu}d\mu=\dfrac{1}{N}(\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x})dx \quad \Rightarrow \int\dfrac{1}{\mu}d\mu=\dfrac{1}{N}(\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x})dx=-\int\dfrac{1}{x}dx$

$\Rightarrow \ln\mu=-\ln x \quad \Rightarrow \mu=\dfrac{1}{x}$

同乘積分因子後可得 $(1+\dfrac{y}{x})dx+\ln x\cdot dy=0$

$\because$ 此為正合 ODE

$\therefore \dfrac{\partial\phi}{\partial x}=1+\dfrac{y}{x} \quad \Rightarrow \phi=x+y\cdot\ln x+f(y)$

$\dfrac{\partial\phi}{\partial y}=\ln|x| \quad \Rightarrow \phi=y\cdot\ln x+g(x)$

比較後可得 $\phi(x,y)=x+y\cdot\ln x=c$

2. 試解: $x^3y'+3x^2y=y^{-3}$

$x^3y'+3x^2y=y^{-3} \;\Rightarrow y^3y'+\dfrac{3}{x}y^4=x^{-3} \longrightarrow$ Bernoulli ODE

令 $u=y^4 \;\Rightarrow \dfrac{du}{dx}=4y^3\cdot\dfrac{dy}{dx}$

$y^3y'+\dfrac{3}{x}y^4=x^{-3} \;\Rightarrow \dfrac{1}{4}\dfrac{du}{dx}+\dfrac{3}{x}u=x^{-3} \;\Rightarrow \dfrac{du}{dx}+\dfrac{12}{x}u=4x^{-3} \longrightarrow$ 一階線性 ODE

$\therefore$ 積分因子為 $\mu=e^{\int\frac{12}{x}dx}=x^{12}$

同乘積分因子可得 $x^{12}u'+12x^{11}u=4x^9$

$\Rightarrow \dfrac{d}{dx}(x^{12}u)=4x^9 \quad \Rightarrow \int d(x^{12}u)=\int 4x^9dx$

$\Rightarrow x^{12}u=\dfrac{2}{5}x^{10}+c \quad \Rightarrow x^{12}y^4=\dfrac{2}{5}x^{10}+c$